Meeting 1:

Motivation:

Let \( \mathbb{Z} \) be set of all integers. Set \( \mathbb{Z} \) with ordinary operations additive (+) and multiplicative (.) has the following properties:

For every \( a, b, c \in \mathbb{Z} \)

I. (i) \( a + b \in \mathbb{Z} \) (closed to additive)
   (ii). \( (a + b) + c = a + (b + c) \) (associative)
   (iii). There is an element 0 in \( \mathbb{Z} \) such that \( a + 0 = 0 + a = a \). The element 0 is called identity element.
   (iv). There exists an element \(-a\) such that \( a + (-a) = -a + a = 0 \). \(-a\) is called inverse of \( a \).
   (v). \( a + b = b + a \). (commutative)

II. (i). \( ab \in \mathbb{Z} \) (closed to multiplicative)
   (ii). \( (ab)c = a(bc) \) (associative)

III. (i). \( a(b + c) = ab + ac \) (left distributive law)
   (ii). \( (a + b)c = ac + bc \) (right distributive law)

We conclude that

I. \( (\mathbb{Z}, +) \) is abelian group
II. \( (\mathbb{Z}, .) \) is semigroup
III. Left distributive and right distributive laws are hold.

Definition 1 (ring):

Let \( R \) be a nonempty set. A Ring \((R, +, .)\) is set \( R \) with two binary operations + and . (called additive and multiplicative) defined on \( R \) such that the following axioms are satisfied:

I. \((R, +)\) is abelian group. For every \( a, b, c \in R \)
   (i). \( (a + b) + c = a + (b + c) \) (associative)
   (ii). There is an element 0 in \( R \) such that \( a + 0 = 0 + a = a \). The element 0 is called identity element.
(iii). There exists an element \(-a\) such that \(a + (-a) = -a + a = 0\). \(-a\) is called inverse of \(a\).

(iv). \(a + b = b + a\). (commutative)

II. \((R, .)\) is semigroup. For every \(a, b, c \in R\)

(i). \((ab)c = a(bc)\) (associative)

III. Left distributive and right distributive laws are hold. For every \(a, b, c \in R\)

(i). \(a(b + c) = ab + ac\) (left distributive law)

(ii). \((a + b)c = ac + bc\) (right distributive law)

Definition 2:

1. If \((R, +, .)\) is ring, then an identity element under additive operation is called zero element, denoted by \(z\).

2. If there exists an element \(u\) such that \(u \neq z\) and \(u\) is identity element under multiplicative operation, then \(u\) is called unity.

3. If ring \(R\) has unity, then ring \(R\) is called ring with unity.

4. If \(u, a \in R\) there exists \(a^{-1} \in R\) such that \(a a^{-1} = a^{-1}a = u\), then \(a\) is called unit.

5. If ring \(R\) is commutative under multiplicative operation, then \(R\) is called commutative ring.

Definition 3:

1. Let \(a\) and \(b\) be nonzero elements of ring \(R\) such that \(ab = z\), then \(a\) are called left zero divisor and \(b\) is called right zero divisor. If \(a\) is left zero divisor and right zero divisor, then \(a\) is called zero divisor.

2. If \(R\) is commutative ring with unity and no zero divisor, then \(R\) is called integral domain.

3. If \(R\) is commutative ring with unity and every nonzero element of \(R\) has inverse under multiplicative, then \(R\) is called field.

Examples:

1. Set of \(\mathbb{Z}\) with ordinary operations additive (+) and multiplicative (\.) is a ring.

2. How about \(\mathbb{R}, \mathbb{Q}, \mathbb{C}\) with ordinary operations additive (+) and multiplicative (\.)?

3. The set \(\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}\) under additive (\(+_4\)) and multiplicative (\(\times_4\)) modulo 4 is a ring.

4. The set \(\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}\) under additive (\(+_5\)) and multiplicative (\(\times_5\)) modulo 5 is a ring.
5. Identify set $Z_n = \{0, 1, 2, \ldots, n-1\}$ under additive ($+_n$) and multiplicative ($\times_n$) modulo $n$ where $n$ is positive integer. Is $Z_n = \{0, 1, 2, \ldots, n-1\}$ ring, integral domain, field? Explain!

6. Identify set $Z_p = \{0, 1, 2, \ldots, p-1\}$ under additive ($+_p$) and multiplicative ($\times_p$) modulo $p$ where $p$ is prime number. Is $Z_p = \{0, 1, 2, \ldots, p-1\}$ ring, integral domain, field? Explain!

7. Let $n$ be positive integer. Is set $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$ with ordinary operations additive (+) and multiplicative (.) a ring, integral domain, field?

8. $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ with additive (+) and multiplicative (x) operations on matrix. Is $N$ a ring, integral domain, field? Explain!

9. $N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ with additive (+) and multiplicative (x) operations on matrix. Is $N$ a ring, integral domain, field? Explain!

10. $K = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$ with additive (+) and multiplicative (x) operations on matrix. Is $K$ a ring, integral domain, field? Explain!