Survey of Linear Transformation Semigroups Whose The Quasi-ideals are Bi-ideals

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Abstract
A sub semigroup \(Q\) of a semigroup \(S\) is called a quasi-ideal of \(S\) if \(SQ \cap QS \subseteq Q\). A sub semigroup \(B\) of a semigroup \(S\) is called a bi-ideal of \(S\) if \(BSB \subseteq B\). For nonempty subset \(A\) of a semigroup \(S\), \((A)_q\) and \((A)_b\) denote respectively the quasi-ideal and bi-ideals of \(S\) generated by \(A\). Let \(BQ\) denote the class of all semigroup whose bi-ideals are quasi-ideals. Let \(M_D\) be a module over a division ring \(D\) and \(\text{Hom}(M_D)\) be a semigroup under composition of all homomorphisms \(\alpha : M_D \to M_D\). The semigroup \(\text{Hom}(M_D)\) is a regular semigroup.

In this paper we will survey which sub semigroups of \(\text{Hom}(M_D)\) whose the quasi-ideals are bi-ideals.

Keywords: Semigroup, \(BQ\), quasi-ideal, bi-ideal

I. Introduction

The notion of quasi-ideal for semigroup was introduced by O. Steinfeld, in 1956. Although bi-ideals are a generalization of quasi-ideals, the notion of bi-ideal was introduced earlier by R.A. Good and D.R. Hughes in 1952.

A sub semigroup \(Q\) of a semigroup \(S\) is called a quasi-ideal of \(S\) if \(SQ \cap QS \subseteq Q\). A sub semigroup \(B\) of a semigroup \(S\) is called a bi-ideal of \(S\) if \(BSB \subseteq B\). Then Quasi-ideals are a generalization of left ideals and right ideals and bi-ideals are a generalization of quasi-ideals.

O. Steinfeld has defined a bi-ideal and quasi-ideal as follows: For nonempty subset \(A\) of a semigroup \(S\), the quasi-ideal \((A)_q\) of \(S\) generated by \(A\) is the intersection of all quasi-ideal of \(S\) containing \(A\) and bi-ideal \((A)_b\) of \(S\) generated by \(A\) is the intersection of all bi-ideal of \(S\) containing \(A\).1)

We use the symbol \(S^1\) to denote a semigroup \(S\) with an identity, otherwise, a semigroup \(S\) with an identity 1 adjoined.4) A.H. Clifford and G.H. Preston in 1) have proved that Proposition 1.1. ([1], page 133) For a nonempty subset \(A\) of a semigroup \(S\),
\[(A)_q = S^1A \cap AS^1 = (SA \cap AS) \cup A\]

Proposition 1.2. ([1], page 133) For a nonempty subset \(A\) of a semigroup \(S\), \[(A)_b = (AS^{-1}A) \cup A = (ASA) \cup A \cup A^2\]

By these definitions, \((A)_q\) and \((A)_b\) are the smallest quasi-ideal and bi-ideal, respectively, of \(S\) containing \(A\). Since every quasi-ideal of \(S\) is a bi-ideal, it follows that for a nonempty subset \(A\) of \(S\), \((A)_b \subseteq (A)_q\). Hence, if \((A)_b\) is a quasi-ideal of \(S\), then \((A)_b = (A)_q\), then we have:

Proposition 1.3. ([1], page 134) If \(A\) is a nonempty subset of a semigroup \(S\) such that \((A)_b \neq (A)_q\), then \((A)_b\) is a bi-ideal of \(S\) which is not a quasi-ideal.

S Lajos has defined a \(BQ\), that is the class of all semigroup whose bi-ideals are quasi-ideals. He has proved that:

Proposition 1.4. ([5], page 238) Every regular semigroup is a \(BQ\)-semigroup.

The next proposition is given by K.M Kapp in 5).

Proposition 1.5. ([5], page 238) Every left [right] simple semigroup and every left [right] 0-simple semigroup is a \(BQ\)-semigroup.

In fact, J. Calais has characterized \(BQ\)-semigroups in 5) as follows
II. Main Results

Let $I_s(M_D)$ be a set of all bijective homomorphisms of $Hom(M_D)$, i.e.:

\[ I_s(M_D) = \{ \alpha \in Hom(M_D) \mid \alpha \text{ is an isomorphism} \} \]

The $I_s(M_D)$ is a regular semigroup, because for all $\alpha \in I_s(M_D)$, there is an $\alpha' = \alpha^{-1} \in I_s(M_D)$ such that $\alpha \circ \alpha' \circ \alpha = \alpha \circ \alpha^{-1} \circ \alpha = \alpha$. By Proposition 1.4, the semigroup $I_s(M_D)$ is in $BQ$.

The next, we construct some subsets of $Hom(M_D)$ as follow:

- $In(M_D) = \{ \alpha \in Hom(M_D) \mid \alpha \text{ is injective} \}$
- $Sur(M_D) = \{ \alpha \in Hom(M_D) \mid \text{Ran } \alpha = V \}$
- $OSur(M_D) = \{ \alpha \in Hom(M_D) \mid \dim(V \setminus \text{Ran } \alpha) \text{ is infinite} \}$
- $OIn(M_D) = \{ \alpha \in Hom(M_D) \mid \dim \text{Ker } \alpha \text{ is infinite} \}$

\[ BHom(M_D) = \{ \alpha \in Hom(M_D) \mid \text{\alpha is injective, } \text{dim}(V \setminus \text{Ran } \alpha) \text{ is infinite} \} \]

By these definitions, so we get: $I_s(M_D) \subseteq In(M_D)$, $I_s(M_D) \subseteq Sur(M_D)$. The $In(M_D)$ and $Sur(M_D)$ are sub semigroups of $Hom(M_D)$:

If $\alpha, \beta \in In(M_D)$, then $\text{Ker } \alpha = \{0\}$, $\text{Ker } \beta = \{0\}$. From this condition, we have, such that $In(M_D)$ is a sub semigroup of $Hom(M_D)$.

If $\alpha, \beta \in Sur(M_D)$, then $\text{Ran } \alpha = V$, $\text{Ran } \beta = V$. From these conditions, we have $V = \text{Ran}(\alpha \circ \beta)$, such that the $Sur(M_D)$ is a sub semigroup of $Hom(M_D)$.

The $OSur(M_D)$ is a semigroup of $Hom(M_D)$, it is caused by:

If $\alpha, \beta \in OSur(M_D)$, so $\dim(V \setminus \text{Ran } \alpha)$ and $\dim(V \setminus \text{Ran } \beta)$ are infinite. For $x \in \text{Ran}(\alpha \circ \beta)$, there is $y \in V$ such that $(y) \alpha \circ \beta = x$ or $(y) \alpha = x$. So, $x \in \text{Ran } \beta$ and we conclude that $\text{Ran}(\alpha \circ \beta) \subseteq \text{Ran } \beta$, so $\dim(V \setminus \text{Ran}(\alpha \circ \beta)$ is infinite.

The $BHom(M_D)$ is a sub semigroup of $Hom(M_D)$:

If $\alpha, \beta \in BHom(M_D)$, then $\alpha, \beta$ are injective and the $\dim(V \setminus \text{Ran } \alpha)$ and $\dim(V \setminus \text{Ran } \beta)$ are infinite. By the previous proofing, so $\dim(V \setminus \text{Ran}(\alpha \circ \beta))$ are infinite and $(\alpha \circ \beta)$ is bijective.

In order to prove that, $In(M_D)$ and $Sur(M_D)$ are in $BQ$ if and only if $\dim V$ is finite, we need this Lemma:

**Lemma 2.1.** ([2], page 407) If $B$ is a basis of $M_D$, $A \subseteq B$ and $\alpha \in Hom(M_D)$ is one-to-one, then

\[ \dim(\text{Ran } (A) \alpha) = |B \setminus A| \]

From this Lemma, so we can prove that:

**Theorem 2.2.** ([2], page 408) $In(M_D) \in BQ$ if and only if $\dim M_D$ is finite.

Proof: If $\dim M_D$ is finite, then $In(M_D) = I_s(M_D)$. So, $In(M_D)$ is a regular semigroup. By Proposition 1.4, $In(M_D) \in BQ$.

The other side, assume that $\dim M_D$ is infinite. Let $B$ be a basis of $M_D$, so $|B|$ is infinite. Let $A = \{ u_n \mid n \in N \}$ is a subset of $B$, where for any distinct $i, j \in N$, $u_i \neq u_j$.

Let $\alpha, \beta, \gamma \in Hom(M_D)$ be defined as follow:

- $(v)\alpha = \begin{cases} u_{2n} & \text{if } v = u_n \text{ for some } n \in N \\ v & \text{if } v \in B \setminus A \end{cases}$
- $(v)\beta = \begin{cases} u_{n+1} & \text{if } v = u_n \text{ for some } n \in N \\ v & \text{if } v \in B \setminus A \end{cases}$
- $(v)\gamma = \begin{cases} u_{n+2} & \text{if } v = u_n \text{ for some } n \in N \\ v & \text{if } v \in B \setminus A \end{cases}$

By this definition, so $\text{ker } \alpha = \text{ker } \beta = \text{ker } \gamma = \{0\}$, such that $\alpha, \beta, \gamma \in In(M_D)$. Next, we have $(u_n)(\beta \circ \alpha) = (u_n)(\alpha \bullet \gamma)$, for all $n \in N$ and for all $v \in B \setminus A$, we have $(v)(\beta \circ \alpha) = (v)(\beta)\alpha = (v)\alpha = v$.
\[ (v(\alpha \circ \gamma)) = ((v\alpha) \gamma) = (v) \gamma = v. \] So we conclude that \( \alpha \neq \beta \circ \alpha = \alpha \circ \gamma. \] By this conditions we have \( \beta \circ \alpha \in \text{In}(M_D)\alpha, \) because \( \beta \in \text{In}(M_D) \) and \( \alpha \circ \gamma \in \alpha \text{In}(M_D) \). We know that \( \beta \circ \alpha = \alpha \circ \gamma, \) so by the Proposition 1.1 we have \( \beta \circ \alpha \in \text{In}(M_D)\alpha \cap a\text{dn}(M_D) = (\alpha)_q \). Suppose that \( \beta \circ \alpha \in (\alpha)_{q}, \) because \( \alpha \neq \beta \circ \alpha, \) by Proposition 1.1 we get \( \beta \circ \alpha \in \text{In}(M_D)\alpha \). Let \( \lambda \in \text{In}(M_D) \) such that \( \beta \circ \alpha = \alpha \circ \lambda \circ \alpha. \) Since \( \alpha \) is injective, so \( \beta = \lambda \circ \alpha. \) Then we have: \( B \setminus \{u_1\} = B\beta = B(\alpha \circ \lambda) = (B\alpha)\lambda = \{B \setminus \{u_{2n-1}\} \mid n \in N\}\lambda. \) By Lemma 2.1, we have: \[ \dim (\text{Ran} \lambda _i \cap \{B \setminus \{u_{2n-1}\} \mid n \in N\}) = \dim (\{B \setminus \{u_{2n-1}\} \mid n \in N\}). \] From these conditions, so this condition is hold: \[ \dim (\text{Ran} \lambda _i \cap \{B \setminus \{u_1\} \}) = \dim (\{B \setminus \{u_{2n-1}\} \mid n \in N\}) \], but in the other hand: \[ \dim (\text{Ran} \lambda _i \cap \{B \setminus \{u_1\} \}) \leq \dim (\{B \setminus \{u_{2n-1}\} \mid n \in N\}) = 1. \] So, there is a contradiction. By Proposition 1.3, we have \( \beta \circ \alpha \notin (\alpha)_b, \) then \( \text{In}(M_D) \notin BQ. \)

**Theorem 2.3.** ([2], page 408) \textit{Sur}(M_D) \in BQ if and only if \( \dim \text{Hom}(M_D) \) is finite.

Proof: The proof of this theorem is similar with the previous theorem.

The other semigroups e.i. \( \text{OSur}(M_D) \) a always belongs to \( BQ \) but is not regular and is neither right 0-simple nor left 0-simple, if \( \dim M_D \) is finite. These condition is guarantied by these propositions:

**Propositions 2.4.** ([2], page 409) The semigroup \( \text{OSur}(M_D) \) isn’t regular.

**Propositions 2.5.** ([2], page 410) The semigroup \( \text{OSur}(M_D) \) is neither right 0-simple nor left 0-simple.

Although \( \text{OSur}(M_D) \) has properties likes above, \( \text{OSur}(M_D) \) is a left ideal of \( \text{Hom}(M_D) \) and is always in \( BQ. \)

For the other semigroup, i.e. \( \text{BHom}(M_D) \) is in \( BQ \) if and only if \( \dim M_D \) is countably infinite. It is caused by this lemma:

**Lemma 2.6.** ([2], page 411) If \( \dim M_D \) is countably infinite, then \( \text{BHom}(M_D) \) is right simple.

The next, we construct the other subsets of \( \text{Hom}(M_D) \):

\[ \text{OInSur}(M_D) \]
\[ = \{ \alpha \in \text{Hom}(M_D) \mid \dim K\alpha, \dim(V \setminus \text{Ran}\alpha) \text{are infinite}\} \]
\[ \text{OBHom}(M_D) \]
\[ = \{ \alpha \in \text{Hom}(M_D) \mid \text{Ran}\alpha = M_D, \dim K\alpha \text{is infinite}\} \]

From the definition, we get:

\[ \text{OInSur}(M_D) = \text{OSur}(M_D) \cap \text{OIn}(M_D), \] this set is not empty set, because \( 0 \in \text{OSur}(M_D) \cap \text{OIn}(M_D) = \text{OInSur}(M_D). \) This set is a sub semigroup of \( \text{Hom}(M_D) \).

**Lemma 2.7.** ([5], page 240) For every infinite dimention of \( M_D, \) \( \text{OInSur}(M_D) \) is a regular sub semigroup of \( \text{Hom}(M_D) \).

The following theorem is the corollary of the previous lemma:

**Theorem 2.8.** ([5], page 240) For every infinite dimention of \( M_D, \) \( \text{OInSur}(M_D) \) is in \( BQ. \)

The set \( \text{OBHom}(M_D) \) is an intersection of \( \text{In}(M_D) \) and \( \text{OSur}(M_D) \). Let \( B \) be a basis of \( M_D \), since \( B \) is infinite, there is a subset \( A \) of \( B \) such that \( |A| = |B\setminus A| = |B| \). Then there exist a bijection \( \varphi : A \to B. \) Define a homomorphisms in \( \text{Hom}(M_D) \) as follow:

\[ (v)(\alpha) = \begin{cases} \varphi(v) & \text{if } v \in A \\ 0 & \text{if } v \in B \setminus A \end{cases} \]

Hence \( \alpha \in \text{OBHom}(M_D) \)

**Lemma 2.9.** ([5], page 241). If \( \dim M_D \) is countably infinite, then \( \text{OBHom}(M_D) \) is left simple.

As the corollary, we get:

**Theorem 2.10.** ([5], page 242). The semigroup \( \text{OBHom}(M_D) \) is in \( BQ \) if and only if \( \dim M_D \) is countably infinite.

### III. Conclusion

From this survey, we can conclude that there is a different conditions such that the sub semigroup of \( \text{Hom}(M_D) \) is in \( BQ, \) i.e:

1. \( \text{In}(M_D) \in BQ \) if and only if \( \dim M_D \) is finite.
2. \( \text{Sur}(M_D) \in BQ \) if and only if \( \dim \text{Hom}(M_D) \) is finite
3. The semigroup \( \text{OSur}(M_D) \) isn’t regular.
4. The semigroup $OSur(M_D)$ is neither right 0-simple nor left 0-simple.
5. If $\dim M_D$ is countably infinite, then $BH\text{Hom}(M_D)$ is right simple.
6. For every infinite dimension of $M_D$, $OlnSur(M_D)$ is a regular sub semigroup of $H\text{om}(M_D)$
7. For every infinite dimension of $M_D$, $OlnSur(M_D)$ is in $BQ$
8. If $\dim M_D$ is countably infinite, then $OBH\text{om}(M_D)$ is left simple.
9. The semigroup $OBH\text{om}(M_D)$ is in $BQ$ if and only if $\dim M_D$ is countably infinite.
10. Finally, we can conclude that not every semigroup in $BQ$ is a regular semigroup and not every semigroup in $BQ$ is either right 0-simple or left 0-simple.

V. References

2. Namnak, C, Kemprasit, Y,"On Semigroups of Linear Transformations whose Bi-ideals are Quasi-ideals”, PU.M.A Vol 12 No4, 405,413 (2001)