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CONSTRUCTING COMPLETE FUZZY RULES OF FUZZY MODEL USING SINGULAR VALUE DECOMPOSITION

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Abstract. *In the fuzzy model, there are many ways to design fuzzy rules from input-output data. Those are gradient descent training, table lookup scheme, recursive least squares and clustering. The aim in this paper is to construct complete fuzzy rules from input-output data using singular value decomposition. Designing fuzzy rules by singular value decomposition is based on minimizing the square of residual between overall output of the real system and identified model. Thus, the designed fuzzy rules are used to construct fuzzy model by choosing fuzzifier, defuzzifier and inference engine. Furthermore, the fuzzy model is applied to predict inflation rate in Indonesia.*

Key words: *fuzzy rules, fuzzy model, singular value decomposition.*

1. Introduction

Designing fuzzy rule base is one of some steps in fuzzy modeling. Fuzzy rule base is the heart of fuzzy model. In recently, fuzzy model was developed by some researcher. Wang, L.X.(1997) created fuzzy model based on table look up scheme, gradient descent training, recursive least squares and clustering methods. The weakness of fuzzy model based on table look up scheme method is that the fuzzy rule base may not be complete so the fuzzy rule can not cover all values in the domain. Wu, T.P. and Chen, S.M. (1999) designed membership function and fuzzy rules from training data using α -cut of fuzzy sets. In Wu and Chen's method, determining membership function and fuzzy rules needs large computations. To reduce complexity in computation, Yam,Y. (1999) built a method to decrease fuzzy rules using singular value decomposition.

Yen, J., *et al*(1998) developed fuzzy model by combining global and local learning to improve the interpretability. New form of fuzzy inference systems that is highly interpretable based on a judicious choice of membership function was presented by Bikdash, M.(1999). Herrera, L.J., *et al*(2005) constructed Takagi-Sugeno-Kang models having high interpretability using Taylor series approximation. Then Pomares, H. *et al*(2002) identified structure of fuzzy systems based on complete rules that can decide which input variables must be taken into account in the fuzzy system and how many membership functions are needed in every selected input variable.

Fuzzy models have been applied to many fields such in economics, engineering, communications, medicine, etc. Specially in economics, forecasting interest rate of inflation in Indonesia by fuzzy model resulted more accuracy than that by regress method(Abadi, 2005 & 2006). Then Abadi(2007) constructed fuzzy time series model using table lookup scheme to forecast Bank Indonesia Certificate and the result gives high accuracy. Based on the previous research, there are interesting topics in fuzzy model especially in determining fuzzy rules that give good prediction accuracy.

In this paper, we design complete fuzzy rules of fuzzy model using singular value decomposition and then its result is used to predict inflation rate in Indonesia. The method is developed using singular value decomposition of firing strength matrix whose entries are only related to the antecedent parts of all fuzzy rules. Accuracy of forecasting depend on choosing the number of singular values. Then we must select the *r* largest singular values to get good prediction accuracy. The proposed method has a higher prediction accuracy than Wang's method in application to forecasting inflation rate.

The rest of this paper is organized as follows. In section 2, we briefly review the definition of singular value decomposition of matrix and its properties. In section 3, we present a method to construct complete fuzzy rules from training data using singular value decomposition. In section 4, we apply the proposed method to forecasting inflation rate. We also compare the proposed method with the Wang's method in the forecasting inflation rate. Finally, some conclusions are discussed in section 5.

2. Singular value decomposition

In this section, we will introduce singular value decomposition of matrix and its properties referred from Scheick, J.T. (1997). Any m x n matrix A can be expressed as

$$A = USV^T \tag{1}$$

where U and V are orthogonal matrices of dimensions m x m, n x n respectively and S is m x n matrix whose entries are 0 except $s_{ii} = \sigma_i$ $i = 1,2,...,r$ with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq 0$, $r \leq \min(m,n)$. Equation (1) is called a singular value decomposition (*SVD*) of A and the numbers σ_i are called singular values of A. If U_i, V_i are columns of U and V respectively, then equation (1) can be written as

$$A = \sum_{i=1}^{r} \sigma_i U_i V_i^T \tag{2}$$

The *SVD* can be used to solve the system $Ax = d$. If A is n x n invertible matrix, then $r = n$ and all $\sigma_i > 0$. Hence $x = A^{-1}d = \sum_{i=1}^{n} \sigma_i^{-1} < d, U_i > V_i$ where $<,>$ is standard inner product in R^n. If A is singular matrix and of arbitrary dimension, then solution of $Ax = d$ is

$$x^+ = \sum_{i=1}^{r} \sigma_i^{-1} < d, U_i > V_i\ . \tag{3}$$

Furthermore $\min\left\{ \| Ax - d \| : x \in F^n \right\} = \| Ax^+ - d \|$. The *SVD* can be used to analysis sensitivity of the system.

Let $\| A \|_F^2 = \sum_{i,j} a_{ij}^2$ be the Frobenius norm of A. Because U and V are orthogonal matrices, then $\| U_i \| = 1$ and $\| V_i \| = 1$. Hence $\| A \|_F^2 = \left\| \sum_{i=1}^{r} \sigma_i U_i V_i^T \right\|_F^2 = \sum_{i=1}^{r} \sigma_i^2$. Let $A = USV^T$ be *SVD* of A. For given $p \leq r$, the optimal rank p approximation of A is given by $A_p = \sum_{i=1}^{p} \sigma_i U_i V_i^T$.

Then $\| A - A_p \|_F^2 = \left\| \sum_{i=1}^{r} \sigma_i U_i V_i^T - \sum_{i=1}^{p} \sigma_i U_i V_i^T \right\|_F^2 = \left\| \sum_{i=p+1}^{r} \sigma_i U_i V_i^T \right\| = \sum_{i=p+1}^{r} \sigma_i^2$. This means that A_p is the best rank p approximation of A and the approximation error depend only on the summation of the square of the rest singular values.

3. Fuzzy model

Fuzzy systems are knowledge based or rule based systems. The heart of a fuzzy system is a knowledge base that is called fuzzy IF-THEN rules.

Let $A_1^i, ..., A_{N_i}^i$ be N_i fuzzy sets that are normal and complete in input domain $U_i \subset R$, i = 1, 2, …, m. Then there are M = $N_1 \times N_2 \times ... \times N_m$ complete fuzzy rules in the following form:

$$R^j\ :\ IF\ x_1 \text{ is } A_{i_1}^1 \text{ and } x_2 \text{ is } A_{i_2}^2 \text{ and...and } x_m\ is\ A_{i_m}^m, THEN\ y \text{ is } B_j \tag{4}$$

where $j = 1,2,..,N_{i_1} \times ... \times N_{i_m}$, $i_s = 1,2,...,N_s$, $s = 1,2,...,m$, input $(x_1, x_2, ..., x_m) \in U_1 \times U_2 \times ... \times U_m \subset R^m$, output $y \in R$ and B_j is normal fuzzy set in R that its center is b_j as free parameter.

If we use singleton fuzzifier, product inference engine, center average defuzzifier and Gaussian membership function, then the fuzzy model constructed using fuzzy rule bases (4) has the following form:

$$y = f(x) = f(x_1, x_2, ..., x_m) = \frac{\sum_{j=1}^{M} b_j \left(\prod_{i=1}^{m} \exp\left(-\left(\frac{x_i - \overline{x}_i^j}{\sigma} \right)^2 \right) \right)}{\sum_{j=1}^{M} \left(\prod_{i=1}^{m} \exp\left(-\left(\frac{x_i - \overline{x}_i^j}{\sigma} \right)^2 \right) \right)}$$

$$= \sum_{j=1}^{M} b_j C_j \tag{5}$$

where σ is real-valued parameter and $C_j = \frac{\prod_{i=1}^{m} \exp\left(-\left(\frac{x_i - \overline{x}_i^j}{\sigma} \right)^2 \right)}{\sum_{j=1}^{M} \left(\prod_{i=1}^{m} \exp\left(-\left(\frac{x_i - \overline{x}_i^j}{\sigma} \right)^2 \right) \right)}$ is the firing strength of rule R^j.

Given P data pairs, we will construct complete fuzzy rules (4) or fuzzy model (5) that minimize the objective function:

$$E = \sum_{k=1}^{P} (\hat{y}(k) - y(k))^2 \tag{6}$$

where $y(k)$ is the output of the real system and $\hat{y}(k)$ is the output of the fuzzy model.

If $Y = (y(1)\ y(2)\ ...y(P))^T \in R^P$, $b = (b_1\ b_2\ ...\ b_M)^T$ and $C = \begin{pmatrix} C_1(1) & C_2(1) & \cdots & C_M(1) \\ C_1(2) & C_2(2) & \cdots & C_M(2) \\ \vdots & \vdots & \vdots & \vdots \\ C_1(P) & C_2(P) & \cdots & C_M(P) \end{pmatrix}$, $C_j(k)$ is value of firing strength of jth rule at kth datum. Then equation (6) can be written as

$$E = (Y - Cb)^T (Y - Cb) \tag{7}$$

where entries of b are free parameters to be determined.

Thus we apply (1) for C, then $C = USV^T$ where U and V are orthogonal matrices of dimensions P x P, M x M respectively and S is P x M matrix whose entries are 0 except $s_{ii} = \sigma_i \quad i = 1, 2, ..., r$ with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq 0$, $r \leq \min(P, M)$. Then (7) will be minimum if $Cb = Y$. The solution of the linear system $Cb = Y$ is

$$b^+ = \sum_{i=1}^{r} \sigma_i^{-1} < Y, U_i > V_i \tag{8}$$

The best rank k approximation of C is $C^{(k)} = \sum_{i=1}^{k} \sigma_i U_i V_i^T$, then $\| C - C^{(k)} \|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$. If $b^{(k)}$ is the solution of $C^{(k)} b = Y$, then we have

$$\begin{aligned}
\sum_{i=1}^{P} (y(i) - y^{(k)}(i))^2 &= \| Y - Y^{(k)} \|_F^2 = \| Cb - C^{(k)} b^{(k)} \|_F^2 \\
&= \left\| \sum_{i=1}^{r} \sigma_i < b, V_i > U_i - \sum_{j=1}^{k} \sigma_j < b^{(k)}, V_j > U_j \right\|_F^2 \\
&= \left\| \sum_{i=1}^{r} \sigma_i < \sum_{t=1}^{r} \frac{< Y, U_t >}{\sigma_t} V_t, V_i > U_i - \sum_{j=1}^{k} \sigma_j < \sum_{s=1}^{k} \frac{< Y, U_s >}{\sigma_s} V_s, V_j > U_j \right\|_F^2 \\
&= \left\| \sum_{i=1}^{r} \sigma_i \frac{< Y, U_i >}{\sigma_i} U_i - \sum_{j=1}^{k} \sigma_j \frac{< Y, U_j >}{\sigma_j} U_j \right\|_F^2 = \left\| \sum_{i=k+1}^{r} < Y, U_i > U_i \right\|_F^2 \\
&\leq \sum_{i=k+1}^{r} \| < Y, U_i > U_i \|_F^2 = \sum_{i=k+1}^{r} \left| < Y, U_i > \right| \| U_i \|_F^2 = \sum_{i=k+1}^{r} \left| < Y, U_i > \right| .
\end{aligned}$$

Therefore if we use $C^{(k)}$ to approximate C, then the error of output model is

$$\sum_{i=1}^{P} (y(i) - y^{(k)}(i))^2 \leq \sum_{i=k+1}^{r} \left| < Y, U_i > \right| \tag{9}$$

This means that the error of training data can be decreased by taking more singular values of C but this may cause increasing error of test data. So we must choose the appropriate r.

Wang (1994) presented a method to generate fuzzy rules. The method is called table lookup scheme. The steps of Wang's method are viewed as follows: (1). Divide the input and output domains into fuzzy regions; (2). Generate fuzzy rules from given data pairs; (3).Compute a degree of each rule; (4). Create a combined fuzzy rule base; (5). Determine output model based on the combined fuzzy rule base.

Fuzzy rules, however, developed by Wang's method may not be complete so fuzzy rules can not cover all values in the domain. Therefore we propose a method to design complete fuzzy rules of fuzzy model from training data using singular value decomposition. The method is presented as follows:

Step1. Define the universes of discourse of input data which fuzzy sets will be defined.
Step2. Define fuzzy sets in each input domains.
Step3. Construct complete fuzzy rules from training data using singular value decomposition.
Step4. Forecast the output using complete fuzzy rules, fuzzifier, fuzzy inference engine and defuzzifier.
Step5. Choose the k largest singular values to minimize the prediction error.

4. Forecasting inflation rate

In this section, we apply the proposed method to forecast inflation rate in Indonesia. The data of inflation rate are taken from January 1999 to January 2003. Data from January 1999 to March 2002 are used to design model and data from April 2002 to January 2003 are used to prediction. The procedure to forecasting inflation rate based on singular value decomposition is given by the following steps:

(1). Define the universe of discourse of each input. In this paper, we will predict inflation rate of month *k* using inflation rate data of months *k*-1 and *k*-2 so we will build fuzzy model using 2-input and 1-output. The universe of discourse of each input is defined as [-2, 4].
(2). Define fuzzy sets on universe of discourse of each input such that fuzzy sets can cover the input spaces. We define thirteen fuzzy sets that are normal and complete on [-2, 4] of each input space with Gaussian membership functions.
(3).Construct 13x13 fuzzy rules:

$$R^j: \text{"IF } x_{k-2} \text{ is } A_{i_1}^1 \text{ and } x_{k-1} \text{ is } A_{i_2}^2 \text{, THEN } x_k \text{ is } B_j\text{"}$$

where B_j is any fuzzy set with its center at b_j that must be determined, j = 1, 2, …, 169, $i_1, i_2 = 1, 2, ..., 13$.

Define firing strength matrix $C = \begin{pmatrix} C_1(1) & C_2(1) & \cdots & C_{169}(1) \\ C_1(2) & C_2(2) & \cdots & C_{169}(2) \\ \vdots & \vdots & \vdots & \vdots \\ C_1(36) & C_2(36) & \cdots & C_{169}(36) \end{pmatrix}$ and apply singular value decomposition of *C*, then $C = \sum_{i=1}^{36} \sigma_i U_i V_i^T$. The singular values of *C* are shown in Figure 1. From Figure 1, there are 36 nonzero singular values. So based on (8), $b^+ = \sum_{i=1}^{36} \sigma_i^{-1} < Y, U_i > V_i$ is a matrix whose entries are the estimation of parameters b_j, *j* = 1, 2, …, 169. So we have 169 complete fuzzy rules.
(4). Design the fuzzy model based on the fuzzy rule base. In this paper, we use singleton fuzzifier, product inference engine, and center average defuzzifier with Gaussian membership function. Choose the *r* largest singular values so that we get the good prediction accuracy measured by mean square error (MSE).

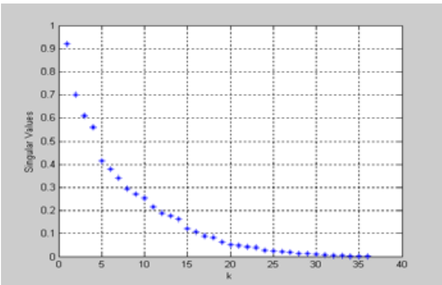

Figure 1. Distribution of singular values of the 36x169 firing strength matrix

Based on Figure 1, there are 36 nonzero singular values of firing strength matrix *C*. If we put all singular values (*r* = 36) to estimate centers of fuzzy sets in the consequent of each rule, then the MSE of training data is 0.00007 but the MSE of test data is 27.778. The MSE value from the different number of singular values can be seen in Table 1. The true and prediction values of inflation rate based on some *r* largest singular values are shown in Figure 2. Based on (9), the fewer we take singular values, the larger we have error of training data. If we take the fewer number of singular values, then the MSE value of test data will reduce until at certain value *r*. It means that we must select the *r* largest singular values so that the MSE of training data and MSE of test data are small.

Table1. Mean square error of taking different number of singular values

MSE	The *r* largest singular values								
	r = 36	*r* =18	*r* = 14	*r* =12	*r* = 10	*r* = 8	*r* = 6	*r* = 4	*r* = 2
MSE of training data	0.00007	0.29615	0.33530	0.38385	0.46270	0.39045	0.47249	0.48529	0.55716
MSE of test data	27.7780	0.25521	0.25404	0.24720	0.25164	0.28513	0.25987	0.25008	0.27772

In this paper, we take only nine different number of singular values to know the properties of MSE of the training and test data. From Table 1, the fuzzy model has a minimal error of test data if we obtain the 12 largest singular values. If we use Wang’s method, then MSE of training data is 0.4578 and MSE of test data is 0.3529. The true and prediction values of inflation rate using Wang’s method are shown in Figure 3.

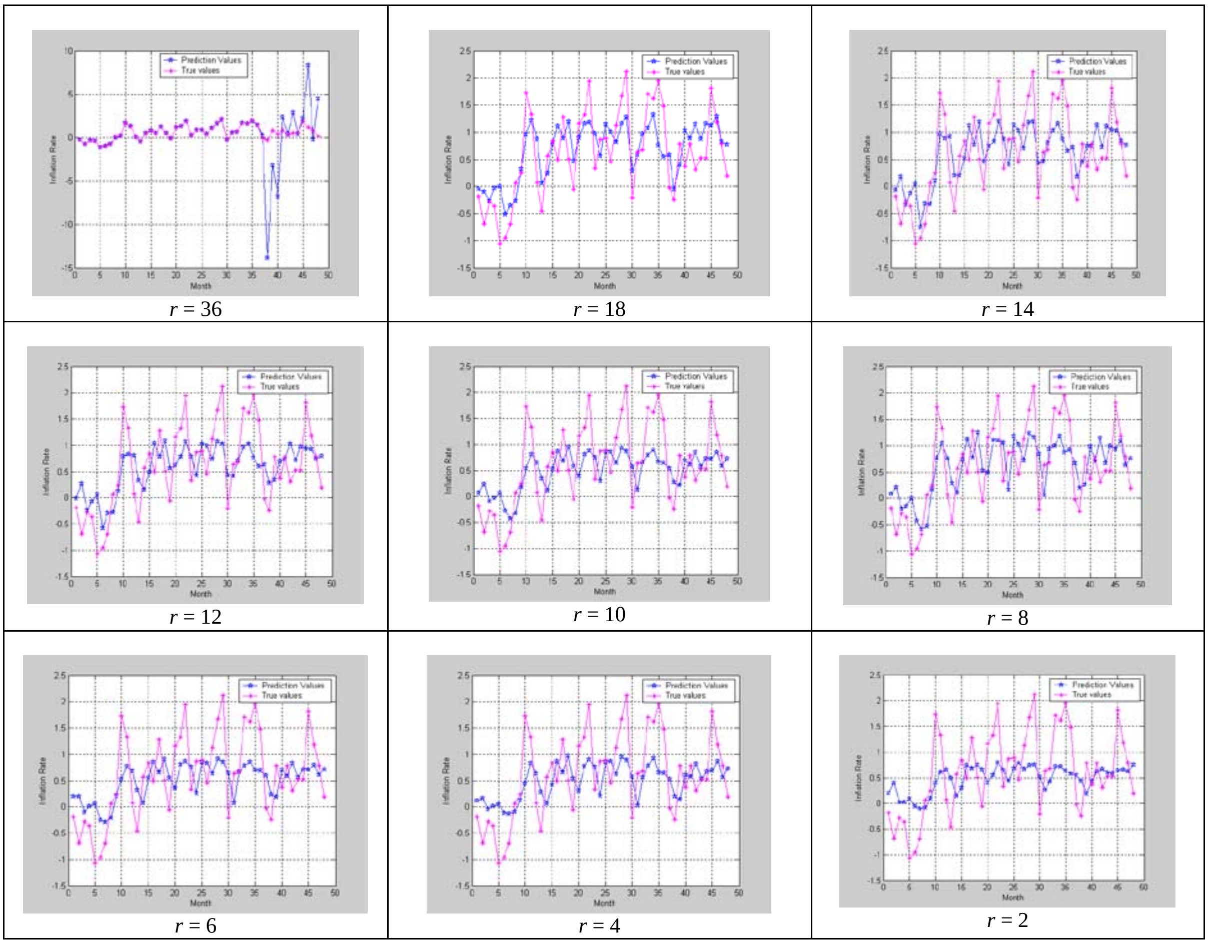

Figure 2. The true and prediction values of inflation rate based on different number of singular values

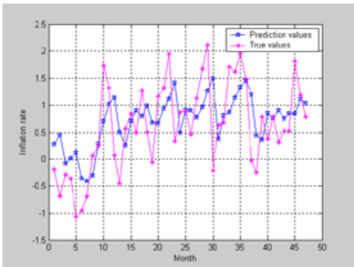

Figure 3. The true and prediction values of inflation rate based on Wang's method

5. Conclusions

In this paper, we have presented a method for designing complete fuzzy rules using singular value decomposition based on *m*-input one-output system. The method is applied to forecasting inflation rate using two-input and the number fuzzy sets defined in each input domain are thirteen. In application to forecasting inflation rate, the proposed method has a higher forecasting accuracy than the Wang's method. The prediction accuracy can be improved using the more fuzzy sets. The number of inputs and kinds of membership functions of fuzzy sets can also influence the accuracy of forecasting so the different number of inputs and kinds of membership functions must be tried to build the fuzzy model. But the increasing number of input is not always followed by the increasing prediction accuracy.

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