2.1.1 Algebraic Properties of $\mathbb{R}$ On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all $a$, $b$ in $\mathbb{R}$ (commutative property of addition);

(A2) $(a + b) + c = a + (b + c)$ for all $a$, $b$, $c$ in $\mathbb{R}$ (associative property of addition);

(A3) there exists an element 0 in $\mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all $a$ in $\mathbb{R}$ (existence of a zero element);

(A4) for each $a$ in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (existence of negative elements);

(M1) $a \cdot b = b \cdot a$ for all $a$, $b$ in $\mathbb{R}$ (commutative property of multiplication);

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a$, $b$, $c$ in $\mathbb{R}$ (associative property of multiplication);

(M3) there exists an element 1 in $\mathbb{R}$ distinct from 0 such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all $a$ in $\mathbb{R}$ (existence of a unit element);

(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$ (existence of reciprocals);

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a$, $b$, $c$ in $\mathbb{R}$ (distributive property of multiplication over addition).
2.1.2 Theorem  (a) If $z$ and $a$ are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$.

(b) If $u$ and $b \neq 0$ are elements in $\mathbb{R}$ with $u \cdot b = b$, then $u = 1$.

(c) If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.

BUKTI

Diketahui $z+a=a$ maka $z=0$

Strategi

$Z=z+0$ (sifat)A3

$Z=Z+a-a$

$Z=a-a$

$Z=0$

(b) Using (M3), (M4), (M2), the assumed equality $u \cdot b = b$, and (M4) again, we get

$$u = u \cdot 1 = u \cdot (b \cdot (1/b)) = (u \cdot b) \cdot (1/b) = b \cdot (1/b) = 1.$$  

(c) We have (why?)

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a.$$
2.1.3 Theorem  (a) If \( a \neq 0 \) and \( b \) in \( \mathbb{R} \) are such that \( a \cdot b = 1 \), then \( b = 1/a \).

(b) If \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

Proof.  (a) Using (M3), (M4), (M2), the hypothesis \( a \cdot b = 1 \), and (M3), we have

\[
\begin{align*}
b &= 1 \cdot b = ((1/a) \cdot a) \cdot b = (1/a) \cdot (a \cdot b) = (1/a) \cdot 1 = 1/a.
\end{align*}
\]

(b) It suffices to assume \( a \neq 0 \) and prove that \( b = 0 \). (Why?) We multiply \( a \cdot b \) by \( 1/a \) and apply (M2), (M4) and (M3) to get

\[
(1/a) \cdot (a \cdot b) = ((1/a) \cdot a) \cdot b = 1 \cdot b = b.
\]

Since \( a \cdot b = 0 \), by 2.1.2(c) this also equals

\[
(1/a) \cdot (a \cdot b) = (1/a) \cdot 0 = 0.
\]

Thus we have \( b = 0 \). Q.E.D.
2.1.4 Theorem \textit{There does not exist a rational number }r\textit{ such that }r^2 = 2.

\textbf{Proof.} Suppose, on the contrary, that }p\textit{ and }q\textit{ are integers such that }\left(\frac{p}{q}\right)^2 = 2\textit{. We may assume that }p\textit{ and }q\textit{ are positive and have no common integer factors other than 1. (Why?) Since }p^2 = 2q^2\textit{, we see that }p^2\textit{ is even. This implies that }p\textit{ is also even (because if }p = 2n - 1\text{ is odd, then its square }p^2 = 2(2n^2 - 2n + 1) - 1\text{ is also odd). Therefore, since }p\textit{ and }q\textit{ do not have 2 as a common factor, then }q\textit{ must be an odd natural number.

Since }p\textit{ is even, then }p = 2m\text{ for some }m \in \mathbb{N},\textit{ and hence }4m^2 = 2q^2\text{, so that }2m^2 = q^2\text{. Therefore, }q^2\textit{ is even, and it follows from the argument in the preceding paragraph that }q\textit{ is an even natural number.

Since the hypothesis that }\left(\frac{p}{q}\right)^2 = 2\textit{ leads to the contradictory conclusion that }q\textit{ is both even and odd, it must be false. Q.E.D.
2.1.5 The Order Properties of $\mathbb{R}$  There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:

(i) If $a, b$ belong to $\mathbb{P}$, then $a + b$ belongs to $\mathbb{P}$.
(ii) If $a, b$ belong to $\mathbb{P}$, then $ab$ belongs to $\mathbb{P}$.
(iii) If $a$ belongs to $\mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}.$$
2.1.4 Theorem  There does not exist a rational number $r$ such that $r^2 = 2$.

Proof. Suppose, on the contrary, that $p$ and $q$ are integers such that $(p/q)^2 = 2$. We may assume that $p$ and $q$ are positive and have no common integer factors other than 1. (Why?) Since $p^2 = 2q^2$, we see that $p^2$ is even. This implies that $p$ is also even (because if $p = 2n - 1$ is odd, then its square $p^2 = 2(2n^2 - 2n + 1) - 1$ is also odd). Therefore, since $p$ and $q$ do not have 2 as a common factor, then $q$ must be an odd natural number.

Since $p$ is even, then $p = 2m$ for some $m \in \mathbb{N}$, and hence $4m^2 = 2q^2$, so that $2m^2 = q^2$. Therefore, $q^2$ is even, and it follows from the argument in the preceding paragraph that $q$ is an even natural number.

Since the hypothesis that $(p/q)^2 = 2$ leads to the contradictory conclusion that $q$ is both even and odd, it must be false. \hfill \Box