2.1.1 Algebraic Properties of $\mathbb{R}$ On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called **addition** and **multiplication**, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all a, b in $\mathbb{R}$ (*commutative property of addition*);

(A2) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$ (*associative property of addition*);

(A3) there exists an element 0 in $\mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all a in $\mathbb{R}$ (*existence of a zero element*);

(A4) for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (*existence of negative elements*);

(M1) $a \cdot b = b \cdot a$ for all a, b in $\mathbb{R}$ (*commutative property of multiplication*);

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{R}$ (*associative property of multiplication*);

(M3) there exists an element 1 in $\mathbb{R}$ *distinct from* 0 such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all a in $\mathbb{R}$ (*existence of a unit element*);

(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$ (*existence of reciprocals*);

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in $\mathbb{R}$ (*distributive property of multiplication over addition*).

2.1.2 Theorem (a) *If z and a are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$*

(b) *If u and $b \neq 0$ are elements in $\mathbb{R}$ with $u \cdot b = b$, then $u = 1$.*

(c) *If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.*

BUKTI

Diketahui z+a=a maka z=0

Strategi

Z=z+0 (sifat)A3

Z=Z+a-a

Z=a-a

Z=0

(b) Using (M3), (M4), (M2), the assumed equality $u \cdot b = b$, and (M4) again, we get

$$u = u \cdot 1 = u \cdot (b \cdot (1/b)) = (u \cdot b) \cdot (1/b) = b \cdot (1/b) = 1.$$

(c) We have (why?)

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a.$$

2.1.3 Theorem (a) *If $a \neq 0$ and b in $\mathbb{R}$ are such that $a \cdot b = 1$, then $b = 1/a$.*

(b) *If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.*

Proof. (a) Using (M3), (M4), (M2), the hypothesis $a \cdot b = 1$, and (M3), we have

$$b = 1 \cdot b = ((1/a) \cdot a) \cdot b = (1/a) \cdot (a \cdot b) = (1/a) \cdot 1 = 1/a.$$

(b) It suffices to assume $a \neq 0$ and prove that $b = 0$. (Why?) We multiply $a \cdot b$ by $1/a$ and apply (M2), (M4) and (M3) to get

$$(1/a) \cdot (a \cdot b) = ((1/a) \cdot a) \cdot b = 1 \cdot b = b.$$

Since $a \cdot b = 0$, by 2.1.2(c) this also equals

$$(1/a) \cdot (a \cdot b) = (1/a) \cdot 0 = 0.$$

Thus we have $b = 0$. Q.E.D.

2.1.4 Theorem *There does not exist a rational number r such that $r^2 = 2$.*

Proof. Suppose, on the contrary, that p and q are integers such that $(p/q)^2 = 2$. We may assume that p and q are positive and have no common integer factors other than 1. (Why?) Since $p^2 = 2q^2$, we see that p^2 is even. This implies that p is also even (because if $p = 2n - 1$ is odd, then its square $p^2 = 2(2n^2 - 2n + 1) - 1$ is also odd). Therefore, since p and q do not have 2 as a common factor, then q must be an odd natural number.

Since p is even, then $p = 2m$ for some $m \in \mathbb{N}$, and hence $4m^2 = 2q^2$, so that $2m^2 = q^2$. Therefore, q^2 is even, and it follows from the argument in the preceding paragraph that q is an even natural number.

Since the hypothesis that $(p/q)^2 = 2$ leads to the contradictory conclusion that q is both even and odd, it must be false. Q.E.D.

2.1.5 The Order Properties of $\mathbb{R}$ There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of **positive real numbers**, that satisfies the following properties:

(i) If a, b belong to $\mathbb{P}$, then $a + b$ belongs to $\mathbb{P}$.

(ii) If a, b belong to $\mathbb{P}$, then ab belongs to $\mathbb{P}$.

(iii) If a belongs to $\mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

2.1.4 Theorem *There does not exist a rational number r such that $r^2 = 2$.*

Proof. Suppose, on the contrary, that p and q are integers such that $(p/q)^2 = 2$. We may assume that p and q are positive and have no common integer factors other than 1. (Why?) Since $p^2 = 2q^2$, we see that p^2 is even. This implies that p is also even (because if $p = 2n - 1$ is odd, then its square $p^2 = 2(2n^2 - 2n + 1) - 1$ is also odd). Therefore, since p and q do not have 2 as a common factor, then q must be an odd natural number.

Since p is even, then $p = 2m$ for some $m \in \mathbb{N}$, and hence $4m^2 = 2q^2$, so that $2m^2 = q^2$. Therefore, q^2 is even, and it follows from the argument in the preceding paragraph that q is an even natural number.

Since the hypothesis that $(p/q)^2 = 2$ leads to the contradictory conclusion that q is both even and odd, it must be false. Q.E.D.