Fuzzy Rules & Fuzzy Reasoning

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Referensi:

Extension Principle & Fuzzy Relations (3.2)

Extension principle

A is a fuzzy set on X:

\[ A = \mu_A(x_1) / x_1 + \mu_A(x_2) / x_2 + \cdots + \mu_A(x_n) / x_n \]

The image of A under f(.) is a fuzzy set B:

\[ B = \mu_B(y_1) / y_1 + \mu_B(y_2) / y_2 + \cdots + \mu_B(y_n) / y_n \]

where \( y_i = f(x_i), \) \( i = 1 \) to \( n \)

If f(.) is a many-to-one mapping, then

\[ \mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x) \]
Extension Principle & Fuzzy Relations (3.2) (cont.)

- Example:

  Application of the extension principle to fuzzy sets with discrete universes

  Let $A = 0.1 / -2 + 0.4 / -1 + 0.8 / 0 + 0.9 / 1 + 0.3 / 2$
  and $f(x) = x^2 - 3$

  Applying the extension principle, we obtain:
  $B = 0.1 / 1 + 0.4 / -2 + 0.8 / -3 + 0.9 / -2 + 0.3 / 1$
  $= 0.8 / -3 + (0.4V0.9) / -2 + (0.1V0.3) / 1$
  $= 0.8 / -3 + 0.9 / -2 + 0.3 / 1$

  where “V” represents the “max” operator

  Same reasoning for continuous universes
Fuzzy relations

- A fuzzy relation \( R \) is a 2D MF:

\[
R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}
\]

- Examples:

Let \( X = Y = \mathbb{IR}^+ \)
and \( R(x, y) = "y \text{ is much greater than } x" \)
The MF of this fuzzy relation can be subjectively defined as:

\[
\mu_R(x, y) = \begin{cases} 
\frac{y - x}{x + y + 2}, & \text{if } y > x \\
0, & \text{if } y \leq x
\end{cases}
\]

if \( X = \{3,4,5\} \) & \( Y = \{3,4,5,6,7\} \)
• Then R can be written as a matrix:

\[
R = \begin{bmatrix}
0 & 0.111 & 0.200 & 0.273 & 0.333 \\
0 & 0 & 0.091 & 0.167 & 0.231 \\
0 & 0 & 0 & 0.077 & 0.143 \\
\end{bmatrix}
\]

where \( R_{i,j} = \mu[x_i, y_j] \)

- \( x \) is close to \( y \) (\( x \) and \( y \) are numbers)
- \( x \) depends on \( y \) (\( x \) and \( y \) are events)
- \( x \) and \( y \) look alike (\( x \) and \( y \) are persons or objects)
- If \( x \) is large, then \( y \) is small (\( x \) is an observed reading and \( Y \) is a corresponding action)
Max-Min Composition

- The max-min composition of two fuzzy relations $R_1$ (defined on $X$ and $Y$) and $R_2$ (defined on $Y$ and $Z$) is

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) \land \mu_{R_2}(y, z)]$$

- Properties:
  - Associativity: $R \circ (S \circ T) = (R \circ S) \circ T$
  - Distributivity over union: $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
  - Weak distributivity over intersection:
    $$R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$$
  - Monotonicity: $S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$
• Max-min composition is not mathematically tractable, therefore other compositions such as max-product composition have been suggested

– Max-product composition

\[
\mu_{R_1 \circ R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) \mu_{R_2}(y, z)]
\]
– Example of max-min & max-product composition

• Let \( R_1 = \text{“x is relevant to y”} \)
  \( R_2 = \text{“y is relevant to z”} \)
be two fuzzy relations defined on \( X \times Y \) and \( Y \times Z \) respectively
\( X = \{1,2,3\}, Y = \{\alpha, \beta, \chi, \delta\} \) and \( Z = \{a,b\} \).

Assume that:

\[
R_1 = \begin{bmatrix}
0.1 & 0.3 & 0.5 & 0.7 \\
0.4 & 0.2 & 0.8 & 0.9 \\
0.6 & 0.8 & 0.3 & 0.2 \\
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
0.9 & 0.1 \\
0.2 & 0.3 \\
0.5 & 0.6 \\
0.7 & 0.2 \\
\end{bmatrix}
\]
The derived fuzzy relation “x is relevant to z” based on $R_1$ & $R_2$

Let’s assume that we want to compute the degree of relevance between $2 \in X$ & $a \in Z$

Using max-min, we obtain:

$$\mu_{R_1 \circ R_2} (2, a) = \max\{0.4 \land 0.9, 0.2 \land 0.8, 0.5 \land 0.9 \land 0.7\}$$

$$= \max\{0.4, 0.2, 0.5, 0.7\}$$

$$= 0.7$$

Using max-product composition, we obtain:

$$\mu_{R_1 \circ R_2} (2, a) = \max\{0.4 \land 0.9, 0.2 \land 0.8, 0.5 \land 0.9 \land 0.7\}$$

$$= \max\{0.36, 0.04, 0.40, 0.63\}$$

$$= 0.63$$
Linguistic Variables

- Conventional techniques for system analysis are intrinsically unsuited for dealing with systems based on human judgment, perception & emotion

- Principle of incompatibility

  • As the complexity of a system increases, our ability to make precise & yet significant statements about its behavior decreases until a fixed threshold

  • Beyond this threshold, precision & significance become almost mutually exclusive characteristics [Zadeh, 1973]
Fuzzy if-then rules (3.3) (cont.)

- The concept of linguistic variables introduced by Zadeh is an alternative approach to modeling human thinking.

- Information is expressed in terms of fuzzy sets instead of crisp numbers.

- **Definition:** A linguistic variable is a quintuple \((x, T(x), X, G, M)\) where:
  
  - \(x\) is the name of the variable
  - \(T(x)\) is the set of linguistic values (or terms)
  - \(X\) is the universe of discourse
  - \(G\) is a syntactic rule that generates the linguistic values
  - \(M\) is a semantic rule which provides meanings for the linguistic values.
Fuzzy if-then rules (3.3) (cont.)

- Example:

  A numerical variable takes numerical values
  \[ \text{Age} = 65 \]

  A linguistic variables takes linguistic values
  \[ \text{Age is old} \]

  A linguistic value is a fuzzy set

  All linguistic values form a term set

  \[ T(\text{age}) = \{ \text{young, not young, very young, ...} \]
  \[ \text{middle aged, not middle aged, ...} \]
  \[ \text{old, not old, very old, more or less old, ...} \]
  \[ \text{not very young and not very old, ...}\} \]
Where each term $T(\text{age})$ is characterized by a fuzzy set of a universe of discourse $X = [0, 100]$
Fuzzy if-then rules (3.3) (cont.)

- The syntactic rule refers to the way the terms in $T(\text{age})$ are generated.

- The semantic rule defines the membership function of each linguistic value of the term set.

- The term set consists of primary terms as (young, middle aged, old) modified by the negation (“not”) and/or the hedges (very, more or less, quite, extremely,…) and linked by connectives such as (and, or, either, neither,…).
Fuzzy if-then rules (3.3) (cont.)

Concentration & dilation of linguistic values

- Let $A$ be a linguistic value described by a fuzzy set with membership function $\mu_A(x)$
  
  $$A^k = \int \frac{[\mu_A(x)]^k}{x}$$
  
  is a modified version of the original linguistic value.

- $A^2 = \text{CON}(A)$ is called the concentration operation

- $\sqrt{A} = \text{DIL}(A)$ is called the dilation operation

- $\text{CON}(A)$ & $\text{DIL}(A)$ are useful in expressing the hedges such as “very” & “more or less” in the linguistic term $A$

- Other definitions for linguistic hedges are also possible
Fuzzy if-then rules (3.3) (cont.)

– Composite linguistic terms

Let’s define:

\[
\text{NOT}(A) = \neg A = \int [1 - \mu_A(x)]/x,
\]

\[
A \text{ and } B = A \cap B = \int [\mu_A(x) \land \mu_B(x)]/x
\]

\[
A \text{ or } B = A \cup B = \int [\mu_A(x) \lor \mu_B(x)]/x
\]

where A, B are two linguistic values whose semantics are respectively defined by \(\mu_A(.)\) & \(\mu_B(.)\)

Composite linguistic terms such as: “not very young”, “not very old” & “young but not too young” can be easily characterized
Example: Construction of MFs for composite linguistic terms

Let's \( \mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + \left(\frac{x}{20}\right)^4} \)

\( \mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6} \)

Where \( x \) is the age of a person in the universe of discourse \([0, 100]\)

- More or less = DIL(old) = \( \sqrt{\text{old}} = \int_x \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6} / x \)
Fuzzy if-then rules (3.3) (cont.)

- Not young and not old = \neg \text{young} \cap \neg \text{old} =
  \int_x \left[ 1 - \frac{1}{1 + \left( \frac{x}{20} \right)^4} \right] \wedge \left[ 1 - \frac{1}{1 + \left( \frac{x-100}{30} \right)^6} \right] / x

- Young but not too young = young \cap \neg \text{young}^2 \text{ (too = very)} =
  \int_x \left[ \frac{1}{1 + \left( \frac{x}{20} \right)^4} \right] \wedge \left[ 1 - \left( \frac{1}{1 + \left( \frac{x}{20} \right)^4} \right)^2 \right] / x

- Extremely old \equiv \text{very very very old} = \text{CON} (\text{CON} (\text{CON} (\text{CON} (\text{old})))) =
  \int_x \left[ \frac{1}{1 + \left( \frac{x-100}{30} \right)^6} \right]^8 / x
(a) Primary Linguistic Values

- Young
- Old

(b) Composite Linguistic Values

- Young but
- Not Too Young
- Not Young and Not Old
- More or Less Old
- Extremely Old

Membership Grades vs. Age

X = age

Range: 0 to 100
Contrast intensification

the operation of contrast intensification on a linguistic value $A$ is defined by

$$
\text{INT}(A) = \begin{cases} 
2A^2 & \text{if } 0 \leq \mu_A(x) \leq 0.5 \\
-2(\neg A)^2 & \text{if } 0.5 \leq \mu_A(x) \leq 1 
\end{cases}
$$

- INT increases the values of $\mu_A(x)$ which are greater than 0.5 & decreases those which are less or equal that 0.5

- Contrast intensification has effect of reducing the fuzziness of the linguistic value $A$
Effects of Contrast Intensifier

Membership Grades

X