CHAPTER 1

Converges in Probability and Distribution

DEFINITION 1. Let $\{X_n\}$, X be random variables. Then $\{X_n\}$ **converges in distribution** to X as $n \to \infty$, written $X_n \to_d X$, if

$$\lim_{n\to\infty} P(X_n \leq X) = \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for each continuity point of the distribution function $F(x)$.

EXAMPLE 2. Let $X_1, X_2, ..., X_n$ be a random sample from a unifrom distribution, $X_i \sim UNIF(0,1)$, and let $Y_n = X_{n:n}$ the largest order statistic. Find limiting distribution of Y_n.

Solution. From equation **??** the CDF of Y_n is $G_{Y_n} = (F_X(y))^n$and

$$F_X(x) = \begin{cases} 0, & 0 \geq x \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

Therefore

$$G_{Y_n}(y) = \begin{cases} 0, & 0 \geq y \\ y^n, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases}$$

and

$$\lim_{n\to\infty} G_{Y_n}(y) = \begin{cases} \lim\limits_{n\to\infty} 0 = 0, & 0 \geq y \\ \lim\limits_{n\to\infty} y^n = 0, & 0 < y < 1 \\ \lim\limits_{n\to\infty} 1 = 1, & y \geq 1 \end{cases}$$

Thus

$$G_Y(y) = \lim_{n\to\infty} G_{Y_n}(y) = \begin{cases} 0 & , y < 1 \\ 1 & , y \geq 1 \end{cases}$$

EXAMPLE 3. Suppose that $X_1, X_2, ..., X_n$ is a random sample from a Pareto distribution, $X_i \sim PAR(1,1)$ or $f_X(x) = (1+x)^{-2}, x > 0$, and let $Y_n = nX_{1:n}$. The CDF of X_i is $F_X(x) = 1 - \frac{1}{1+x}, x > 0$, Find limiting distribution of Y_n

Solution. From equation **??**,

$$G_{Y_n}(y) = \begin{cases} 1 - \left(1 - F_X\left(\frac{y}{n}\right)\right)^n = 1 - \left(\frac{1}{1+\frac{y}{n}}\right)^n = 1 - \left(1 + \frac{y}{n}\right)^{-n} & , 0 < y \\ 0 & , y \leq 0 \end{cases}$$

and

$$G_Y(y) = \lim_{n\to\infty} G_{Y_n}(y) = \begin{cases} \lim\limits_{n\to\infty} 1 - \left(1 + \frac{y}{n}\right)^{-n} = 1 - e^{-y} & , 0 < y \\ 0 & , y \leq 0 \end{cases}$$

EXAMPLE 4. Rework Example 3 for $Y_n = X_{n:n}$

Solution. From equation ??,

$$G_{Y_n}(y) = \begin{cases} (F_X(y))^n = \left(1 - \left(\frac{1}{1+y}\right)\right)^n & ,0 > y \\ 0 & ,y \geq 0 \end{cases}$$

and

$$G_Y(y) = \lim_{n\to\infty} G_{Y_n}(y) = \begin{cases} \lim\limits_{n\to\infty} \left(\frac{y}{1+y}\right)^n = 0 & ,0 > y \\ 0 & ,y \geq 0 \end{cases}$$

Thus Y_n does not have limit distribution.

DEFINITION 5. The function $G(y)$ is the CDF of degenerate distribution at the value $y = c$ if

$$G_Y(y) = \begin{cases} 0 & ,y < c \\ 1 & ,y \geq c \end{cases}$$

In other words, $G(y)$ is the CDF of discrete distribution that assigns probability one at the value $y = c$ and zero otherwise.

EXAMPLE 6. Let $X_1, X_2, ..., X_n$ is a random sample from an Exponensial distribution, $X_i \sim EXP(\theta)$ or $f_X(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0$, and let $Y_n = X_{1:n}$

Solution. It follows that the CDF of Y_n is

$$G_{Y_n}(y) = \begin{cases} 1 - e^{-\frac{ny}{\theta}} & ,y > 0 \\ 0 & ,y \leq 0 \end{cases}$$

Then

$$G_Y(y) = \lim_{n\to\infty} G_{Y_n}(y) = \begin{cases} \lim\limits_{n\to\infty} 1 - e^{-\frac{ny}{\theta}} = 1 & ,0 < y \\ 0 & ,y \leq 0 \end{cases}$$

which is corresponds to a degenerate distribution at the value $y = 0$.

DEFINITION 7. A sequence of random variables $Y_1, Y_2, ...$ is said to **convergence stochastically** to a constant c, written $Y_n \to_{stochastic} c$, if it has a limiting distribution that is degenerate at $y = c$.

EXAMPLE 8. For Example 2 and from Definition 5 and Definition 7, and $G(y)$ is the CDF of degenerate distribution at the value $y = 1$ and $Y_n \to_{stochastic} 1$.

THEOREM 9. ***(Continuity Theorem)*** *Let $Y_1, Y_2, ...$ be a sequence of random variables with respective CDF's $G_{Y_1}(y), G_{Y_2}(y), ...$ and MGF's $M_{Y_1}(t), M_{Y_2}(t), ...$ If $M_Y(t)$ is the MGF of a CDF $G_Y(y)$, and if $\lim\limits_{n\to\infty} M_{Y_n}(t) = M_Y(t)$ for all t in open interval containing zero, $-h < t < h$, then $\lim\limits_{n\to\infty} G_{Y_n}(y) = G_Y(y)$ for all continuity points of $G_Y(y)$.*

In other words,

$$\lim_{n\to\infty} M_{Y_n}(t) = M_Y(t) \Rightarrow \lim_{n\to\infty} G_{Y_n}(y) = G_Y(y) \Rightarrow Y_n \to_d Y$$

EXAMPLE 10. Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution, $X_i \sim BIN(1, p)$, and consider $Y_n = \sum\limits_{i=1}^{n} X_i$.

Solution. Let $np = \mu$ for fixed $\mu > 0$ then $p \to 0$ as $n \to \infty$. Thus, from Theorem **??**,

$$\begin{aligned} M_{Y_n}(t) &= \left(pe^t + q\right)^n \\ &= \left(\frac{\mu e^t}{n} + 1 - \frac{\mu}{n}\right)^n \\ &= \left(1 + \frac{\mu\left(e^t - 1\right)}{n}\right)^n \end{aligned}$$

And

$$\lim_{n\to\infty} M_{Y_n}(t) = \lim_{n\to\infty}\left(1 + \frac{\mu\left(e^t - 1\right)}{n}\right)^n = e^{\mu\left(e^t - 1\right)}$$

Since $M_Y(t) = e^{\mu\left(e^t-1\right)}$ is the MGF of the Poisson distribution with mean μ. Thus, $Y_n \to_d Y \sim POI(\mu = np)$.

EXAMPLE 11. Let $X_1, X_2, ..., X_n$ be a random sample and consider $Y_n = \sum\limits_{i=1}^{n} X_i$ and $Z_n = \frac{Y_n - np}{\sqrt{npq}}$.

Solution. Let $\sigma_n = \sqrt{npq}$, $Z_n = \frac{Y_n}{\sigma_n} - \frac{np}{\sigma_n}$. Since $Z_n = \frac{Y_n}{\sigma_n} - \frac{np}{\sigma_n}$, then $M_{Z_n}(t) = M_{\frac{Y_n}{\sigma_n} - \frac{np}{\sigma_n}}(t) = e^{-\frac{npt}{\sigma_n}} M_{Y_n}\left(\frac{t}{\sigma_n}\right)$. Therefore, using the series expansion $e^a = 1 + a + \frac{a^2}{2} + ...$,

$$\begin{aligned} M_{Z_n}(t) &= e^{-\frac{npt}{\sigma_n}}\left(pe^{\frac{t}{\sigma_n}} + q\right)^n \\ &= \left[e^{-\frac{pt}{\sigma_n}}\left(pe^{\frac{t}{\sigma_n}} + q\right)\right]^n \\ &= \left[\left(1 - \frac{pt}{\sigma_n} + \frac{p^2t^2}{2\sigma_n^2} + ...\right)\left(p(1 + \frac{t}{\sigma_n} + \frac{t^2}{2\sigma_n^2} + ...) + q\right)\right]^n \\ &= \left[\left(1 - \frac{pt}{\sigma_n} + \frac{p^2t^2}{2\sigma_n^2} + ...\right)\left(1 + \frac{pt}{\sigma_n} + \frac{pt^2}{2\sigma_n^2} + ...\right)\right]^n \\ &= \left[\left(1 + \frac{t^2}{2n} + \frac{d(n)}{n}\right)\right]^n \end{aligned}$$

where $d(n) \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} M_{Z_n}(t) = \lim_{n\to\infty}\left[\left(1 + \frac{t^2}{2n} + \frac{d(n)}{n}\right)\right]^n = e^{\frac{t^2}{2}}$$

In other words, $Z_n = \frac{Y_n - np}{\sqrt{npq}} \to_d Z \sim N(0, 1)$.

EXAMPLE 12. Let $Z_1, Z_2, ..., Z_n$ be a random sample and $Z_i \sim N(0, 1)$, Find the limiting distribution of $Z_n = \frac{\sum\limits_{i=1}^{n} Z_i + \frac{1}{n}}{\sqrt{n}}$.

Solution. Since MGF of Z_i is $M_Z(t) = e^{\frac{1}{2}t^2}$, then from Theorem **??** MGF of Z_n is,

$$
\begin{aligned}
M_{Z_n}(t) &= E\left(e^{Z_n t}\right) \\
&= E\left(e^{\left(\frac{\sum_{i=1}^{n} Z_i + \frac{1}{n}}{\sqrt{n}}\right)t}\right) \\
&= E\left(e^{\frac{t}{\sqrt{n}} + \frac{t}{\sqrt{n}}\sum_{i=1}^{n} Z_i}\right) \\
&= e^{\frac{t}{\sqrt{n}}} E\left(e^{\frac{t}{\sqrt{n}}Z_1} e^{\frac{t}{\sqrt{n}}Z_2} \ldots e^{\frac{t}{\sqrt{n}}Z_n}\right) \\
&= e^{\frac{t}{\sqrt{n}}}\left(M_{Z_n}\left(\frac{t}{\sqrt{n}}\right)\right)^n
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} M_{Z_n}(t) = \lim_{n\to\infty} e^{\frac{t}{\sqrt{n}}}\left(e^{\frac{t^2}{2n}}\right)^n = \lim_{n\to\infty} e^{\frac{t}{\sqrt{n}}} e^{\frac{t^2}{2}} = e^{\frac{t^2}{2}}
$$

Thus, $Z_n \to_d Z \sim N(0,1)$.