CHAPTER 1

Converges in Probability and Distribution

**Definition 1.** Let \( \{X_n\} \), \( X \) be random variables. Then \( \{X_n\} \) converges in distribution to \( X \) as \( n \to \infty \), written \( X_n \to_d X \), if

\[
\lim_{n \to \infty} P(X_n \leq x) = \lim_{n \to \infty} F_{X_n}(x) = F_X(x)
\]

for each continuity point of the distribution function \( F(x) \).

**Example 2.** Let \( X_1, X_2, \ldots, X_n \) be a random sample from a uniform distribution, \( X_i \sim UNIF(0, 1) \), and let \( Y_n = X_{n:n} \) the largest order statistic. Find limiting distribution of \( Y_n \).

Solution. From equation ?? the CDF of \( Y_n \) is \( G_{Y_n} = (F_X(y))^n \) and

\[
F_X(x) = \begin{cases} 
0, & 0 \geq x \\
x, & 0 < x < 1 \\
1, & x \geq 1 
\end{cases}
\]

Therefore

\[
G_{Y_n}(y) = \begin{cases} 
0, & 0 \geq y \\
y^n, & 0 < y < 1 \\
1, & y \geq 1 
\end{cases}
\]

and

\[
\lim_{n \to \infty} G_{Y_n}(y) = \begin{cases} 
\lim_{n \to \infty} 0 = 0, & 0 \geq y \\
\lim_{n \to \infty} y^n = 0, & 0 < y < 1 \\
\lim_{n \to \infty} 1 = 1, & y \geq 1 
\end{cases}
\]

Thus

\[
G_Y(y) = \lim_{n \to \infty} G_{Y_n}(y) = \begin{cases} 
0, & y \leq 1 \\
1, & y > 1 
\end{cases}
\]

**Example 3.** Suppose that \( X_1, X_2, \ldots, X_n \) is a random sample from a Pareto distribution, \( X_i \sim PAR(1, 1) \) or \( f_X(x) = (1 + x)^{-2}, x > 0 \), and let \( Y_n = nX_{1:n} \). The CDF of \( X_i \) is \( F_X(x) = 1 - \frac{1}{1+x}, x > 0 \). Find limiting distribution of \( Y_n \).

Solution. From equation ??,

\[
G_{Y_n}(y) = \begin{cases} 
1 - (1 - F_X\left(\frac{y}{n}\right))^n = 1 - \left(\frac{1}{1+y}\right)^n = 1 - (1 + \frac{y}{n})^{-n}, & 0 < y \\
0, & y \leq 0 
\end{cases}
\]

and

\[
G_Y(y) = \lim_{n \to \infty} G_{Y_n}(y) = \begin{cases} 
\lim_{n \to \infty} 1 - (1 + \frac{y}{n})^{-n} = 1 - e^{-y}, & 0 < y \\
0, & y \leq 0 
\end{cases}
\]
Example 4. Rework Example 3 for $Y_n = X_{n:n}$.

Solution. From equation ??,

$$G_{Y_n}(y) = \begin{cases} (F_X(y))^n = (1 - \left(\frac{1}{1+y}\right))^n, & 0 > y \\ 0, & y \geq 0 \end{cases}$$

and

$$G_Y(y) = \lim_{n \to \infty} G_{Y_n}(y) = \begin{cases} \lim_{n \to \infty} \left(\frac{y}{1+y}\right)^n = 0, & 0 > y \\ 0, & y \geq 0 \end{cases}$$

Thus $Y_n$ does not have limit distribution.

Definition 5. The function $G(y)$ is the CDF of degenerate distribution at the value $y = c$ if

$$G_Y(y) = \begin{cases} 0, & y < c \\ 1, & y \geq c \end{cases}$$

In other words, $G(y)$ is the CDF of discrete distribution that assigns probability one at the value $y = c$ and zero otherwise.

Example 6. Let $X_1, X_2, ..., X_n$ is a random sample from an Exponential distribution, $X_i \sim \text{EXP}(\theta)$ or $f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, x > 0$, and let $Y_n = X_{1:n}$.

Solution. It follows that the CDF of $Y_n$ is

$$G_{Y_n}(y) = \begin{cases} 1 - e^{-\frac{ny}{\theta}}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

Then

$$G_Y(y) = \lim_{n \to \infty} G_{Y_n}(y) = \begin{cases} \lim_{n \to \infty} 1 - e^{-\frac{ny}{\theta}} = 1, & 0 < y \\ 0, & y \leq 0 \end{cases}$$

which corresponds to a degenerate distribution at the value $y = 0$.

Definition 7. A sequence of random variables $Y_1, Y_2, ...$ is said to converge stochastically to a constant $c$, written $Y_n \to_{\text{stochastic}} c$, if it has a limiting distribution that is degenerate at $y = c$.

Example 8. For Example 2 and from Definition 5 and Definition 7, and $G(y)$ is the CDF of degenerate distribution at the value $y = 1$ and $Y_n \to_{\text{stochastic}} 1$.

Theorem 9. (Continuity Theorem) Let $Y_1, Y_2, ...$ be a sequence of random variables with respective CDF’s $G_{Y_1}(y), G_{Y_2}(y), ...$ and MGF’s $M_{Y_1}(t), M_{Y_2}(t), ...$. If $M_Y(t)$ is the MGF of a CDF $G_Y(y)$, and if $\lim_{n \to \infty} M_{Y_n}(t) = M_Y(t)$ for all $t$ in open interval containing zero, $-h < t < h$, then $\lim_{n \to \infty} G_{Y_n}(y) = G_Y(y)$ for all continuity points of $G_Y(y)$.

In other words,

$$\lim_{n \to \infty} M_{Y_n}(t) = M_Y(t) \Rightarrow \lim_{n \to \infty} G_{Y_n}(y) = G_Y(y) \Rightarrow Y_n \to_{\text{d}} Y$$

Example 10. Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution, $X_i \sim \text{BIN}(1, p)$, and consider $Y_n = \sum_{i=1}^{n} X_i$. 
1. CONVERGENCE IN PROBABILITY AND DISTRIBUTION

Solution. Let \( np = \mu \) for fixed \( \mu > 0 \) then \( p \to 0 \) as \( n \to \infty \). Thus, from Theorem ??,

\[
M_{Y_n}(t) = \left( pe^t + q \right)^n = \left( \frac{\mu e^t}{n} + 1 - \frac{\mu}{n} \right)^n = \left( 1 + \frac{\mu(e^t-1)}{n} \right)^n
\]

And

\[
\lim_{n \to \infty} M_{Y_n}(t) = \lim_{n \to \infty} \left( 1 + \frac{\mu(e^t-1)}{n} \right)^n = e^{\mu(e^t-1)}
\]

Since \( M_Y(t) = e^{\mu(e^t-1)} \) is the MGF of the Poisson distribution with mean \( \mu \). Thus, \( Y_n \to_d Y \sim \text{POI}(\mu = np) \).

**Example 11.** Let \( X_1, X_2, \ldots, X_n \) be a random sample and consider \( Y_n = \sum_{i=1}^{n} X_i \) and \( Z_n = \frac{Y_n - np}{\sqrt{npq}} \).

Solution. Let \( \sigma_n = \sqrt{npq} \). \( Z_n = \frac{Y_n - np}{\sigma_n} \). Since \( Z_n = \frac{Y_n - np}{\sigma_n} \), then \( M_{Z_n}(t) = M_{Y_n}(\frac{t}{\sigma_n}) \). Therefore, using the series expansion \( e^a = 1 + a + \frac{a^2}{2} + \ldots \),

\[
M_{Z_n}(t) = e^{-\frac{np}{\sigma_n}} \left( pe^{\frac{t}{\sigma_n}} + q \right)^n
\]

\[
= \left[ 1 + \frac{pt}{\sigma_n} + \frac{p^2 t^2}{2! \sigma_n^2} + \ldots \right] \left( 1 + \frac{q}{\sigma_n} + \frac{q^2}{2! \sigma_n^2} + \ldots \right)^n
\]

\[
= \left[ 1 + \frac{t^2}{2n} + \frac{d(n)}{n} \right]^n
\]

where \( d(n) \to 0 \) as \( n \to \infty \). Thus,

\[
\lim_{n \to \infty} M_{Z_n}(t) = \lim_{n \to \infty} \left[ 1 + \frac{t^2}{2n} + \frac{d(n)}{n} \right]^n = e^{t^2}
\]

In other words, \( Z_n = \frac{Y_n - np}{\sqrt{npq}} \to_d Z \sim N(0,1) \).

**Example 12.** Let \( Z_1, Z_2, \ldots, Z_n \) be a random sample and \( Z_i \sim N(0,1) \). Find the limiting distribution of \( Z_n = \frac{\sum_{i=1}^{n} Z_i + \frac{1}{n}}{\sqrt{n}} \).
Solution. Since MGF of $Z_i$ is $M_Z(t) = e^{\frac{1}{2}t^2}$, then from Theorem ?? MGF of $Z_n$ is,

\[ M_{Z_n}(t) = E \left( e^{Z_n t} \right) = e^{E\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i^2\right)} = e^{\frac{1}{\sqrt{n}}E\left(\sum_{i=1}^{n} Z_i^2\right)} = e^{\frac{1}{\sqrt{n}}M_{Z_n}\left(\frac{1}{\sqrt{n}}\right)^n} \]

Therefore,

\[ \lim_{n \to \infty} M_{Z_n}(t) = \lim_{n \to \infty} e^{\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\right)^n} = \lim_{n \to \infty} e^{\frac{1}{\sqrt{n}}} e^{\frac{1}{\sqrt{n}}} = e^{\frac{1}{2}} \]

Thus, $Z_n \to_d Z \sim N(0, 1)$. 