CHAPTER 1

Combinatorial Analysis

1.1. The Basic Pinciple of Counting

A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals—and will be called functional—as long as no two consecutive antennas are defective. If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional? For instance, in the special case where n = 4 and m = 2, there are 6 possible system configurations, namely,

Counting the number of ways in which an **event** may occur can be a tedious porblem in complicated **experiment**. A few helpful counting technique will be discussed.

EXAMPLE 1. How many ways can a 20-questions true-false be answered?

Solution.

EXAMPLE 2. (a) How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers? (b) And how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution.

THEOREM 3. *If the are N possible outcomes of each of r trials of an experiment, then there are N^r possible outcomes in the sample space.*

1.2. Permutations

An ordered arrangement of a set of objects is known as a **permutation**.

EXAMPLE 4. How many number of arrangements are possible for three distinct letters A, B, C ?

Solution.

THEOREM 5. *The number of permutations of n distinguishable objects is $n!$*

PROOF. This follows by applying the multiplication principle. To fill n positions with n distinct objects, the first position may be filled n ways using any one of the n objects, the second position may be filled $n-1$ ways using any remaining $n-1$ objects, and so on until the last object is placed in the last position. Thus by the multiplication principle, this operation may be carried out in $n.(n-1).....1 = n!$ ways. □

One also may be interested in the number of ways of selecting r objects from n distinct objects and then ordering these r objects.

EXAMPLE 6. How many different arrangements are possible if five distinct letters A, B, C, D, E taken three at a time?

Solution.

THEOREM 7. *The number of permutations of n distinct objects taken r at time is*

$$P_{n,r} = \frac{n!}{(n-r)!}$$

EXAMPLE 8. If three distinct letters A, B, C are randomly arranged in a circle, how many different arrangements are possible?

Solution.

THEOREM 9. *The number of permutations of n distinct objects arranged in a circle is*

$$(n-1)!$$

EXAMPLE 10. How many different letter arrangements can be formed from the letters EYE?

Solution.

The concept can be generalized to the case of permuting k types of objects.

THEOREM 11. *The number of permutations of n objects of which r_1 are of one kind, r_2 are of a second kind, ..., r_k of a kth kind is*

$$\frac{n!}{r_1!r_2!...r_k!}$$

EXAMPLE 12. How many different letter arrangements can be formed from the letters $PEPPER$?

Solution.

EXAMPLE 13. If 4 different popsicles are to be distributed equally among two children, how many divisions are possible?

Solution.

THEOREM 14. *The number of ways of partitioning a set of n objects into k cells with r_1 objects in the first cell, r_2 objects in the second cell, and so forth is*

$$\left(\begin{array}{c} n \\ r_1, r_2, ...r_k \end{array} \right) = \frac{n!}{r_1!r_2!...r_k!}$$

where $r_1 + r_2 + ... + r_k = n$.

Partitioning assumes that the number of objects to be placed in each cell is fixed, and that the order in which the object are placed into cells is not considered.

EXAMPLE 15. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

Solution.

1.3. Combinations

If the order of the objects is not important, then one may simply be interested in the number of **combinations** that are possible when selecting r objects from n distinct objects.

EXAMPLE 16. If five cards are drawn from a deck of card without replecement, how many five-card hands are there?

Solution.

THEOREM 17. *The number of combinations of* n *distinct objects chosen* r *at a time is*

$$C_{n,r} = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{(n-r)!r!}$$

EXAMPLE 18. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution.

The value of $\left(\begin{array}{c} n \\ r \end{array} \right)$ are often referred to as binomial coefficients because of their prominence in the binomial theorem.

THEOREM 19. $(x+y)^n = \sum\limits_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k}$

PROOF. Consider the product

$$(x_1+y_1)(x_2+y_2)\dots(x_n+y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, ..., n$. For example,

$$(x_1+y_1)(x_2+y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

Now, how many of the 2^n terms in the sum will have k of the $x_i' s$ and $(n-k)$ of the $y_i' s$ as factors? As each term consisting of k of the $x_i' s$ and $(n-k)$ of the $y_i' s$ corresponds to a choice of a group of k from the n values $x_1, x_2, ..., x_n$, there are $\left(\begin{array}{c} n \\ k \end{array} \right)$ such terms. Thus, letting $x_i = x$, $y_i = y$, $i = 1, 2, ..., n$, we see that $(x+y)^n = \sum\limits_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k}$ □

1.4. Problems

(1) Code words are formed from the letters A through Z.
   (a) How many 26-letter words can be formed without repeating any letters?
   (b) How many 5-letter words can be formed without repeating any letters?
   (c) How many 5-letter words can be formed if letters can be repeated?
(2) License plate numbers consist of two letters followed by a four-digit number, such as RO5174.

   (a) How many different plates are possible if letters and digits can be repeated?
   (b) Answer (a) if letters can be repeated but digit cannot.
   (c) How many of the plates in (b) have a four-digit number that is greater than 5500?

(3) In how many ways can three boys and three girls sit in row if boys and girls must alternate?

(4) How many odd three-digit numbers can be formed from the digits 0, 1, 2, 3, 4 if digits can be repeated, but the first digit cannot be zero?

(5) A club consists of 17 men and 13 women, a committe of five members must be chosen.
   (a) How many committees are possible?
   (b) How many committees are possible with three men and two women?
   (c) Answer (b) if a particular man must be included.

(6) A football coach has 49 players available for duty on a special kick-receiving team. If 11 must be chosen to play on this special team, how many different teams are possible?

(7) Seven people show up to apply for jobs as cashiers.
   (a) If only three jobs are available, how many ways can three be selected from the seven applicants?
   (b) In how many different ways could the seven applicants be lined up while waiting for an interview?
   (c) If there are four females and three males, in how many ways can the applicants be line up if the first three are females?

(8) The club in exercise 5 must elect three officers: presidents, vice-presidents, secretary. How many different ways can this turn out?

(9) How many ways can 10 students be lined up to get on a bus if a particular pair of students refuse to follow each other in line?

(10) A kindergarten student has 12 crayons.
   (a) How many ways can three blue, four red, and five green crayons be arranged in a row?
   (b) How many ways can 12 distinct crayons be placed in three boxes containing 3, 4, 5 crayons, respectively?

(11) How many ways can you partition 26 letters into boxes containing 9, 11, and 6 letters?

(12) How many ways can you permute 9 a's, 11 b's, and 6 c's?

(13) A contest consists of finding all of the code words that can be formed from the letters in the name RAHMAT. Assume that the letter A can be used twice, but the others at most once.
   (a) How many five-letter words can be formed?
   (b) How many two-letter words can be formed?
   (c) How many words can be formed?

(14) Three buses are available to transport 60 students on a field trip. The buses seat 15, 20, 25 passengers, respectively. How many different ways can the students be loaded on the buses?

(15) A certain machine has nine switches mounted in a row. Each switch has three positions, a, b, and c.
   (a) How many different settings are possible?

(b) Answer (a) if each position is used three times

(16) Suppose the winning number in a lottery is a four-digit number determined by drawing four slips of paper (without replacement) from a box that contains nine slips numbered consecutively 1 through 9 and then recording the digits in order from the smallest to largest.

(a) How many different lottery numbers are possible?

(b) How many different lottery numbers are possible if the digits are recorded in the order they were drawn?