Valid and Invalid Arguments

An argument is a sequence of statements such that
• all statements but the last are called *hypotheses*
• the final statement is called the *conclusion.*
• the symbol \( \therefore \) read “therefore” is usually placed just before the conclusion.

Example:

\[
\begin{align*}
p \land \sim q & \rightarrow r \\
p \lor q \\
q & \rightarrow p \\
\therefore & \quad r
\end{align*}
\]

An argument is said to be *valid* if - whenever all hypotheses are true, the conclusion must be true.
Example of a valid argument (form)

\[ p \land (q \lor r) \]
\[ \sim q \]
\[ \therefore p \land r \]

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<th>p</th>
<th>q</th>
<th>r</th>
<th>( p \land (q \lor r) )</th>
<th>( \sim q )</th>
<th>( p \land r )</th>
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11/26/2013
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An invalid argument

\[ p \rightarrow q \lor \sim r \]
\[ q \rightarrow p \lor r \]
\[ \therefore p \rightarrow r \]

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Invalid row

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Tautology
- is a statement (form) that is always true regardless of the truth values of the individual statement variables.

Examples:
• $p \lor \sim p$ (eg. the number $n$ is either $> 0$ or $\leq 0$)
• $p \land q \rightarrow p$
• $(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow r)$

We need to study tautologies because any valid argument is equivalent to a tautology. In particular, every theorem we have proved is a tautology.
Example:

\[ p \land (q \lor r) \]
\[ \sim q \]
\[ \therefore p \land r \]

is a valid argument,

In other words, an argument

\[ H_1 \]
\[ H_2 \]
[...]
\[ H_n \]
\[ \therefore \text{Conclusion} \]

is valid if and only if

\[ H_1 \land H_2 \land \ldots \land H_n \rightarrow \text{conclusion} \]

is a tautology.

\[ [p \land (q \lor r)] \land [\sim q] \rightarrow [p \land r] \]

is a tautology.
Two most important valid argument forms

**Modus Ponens**: means method of affirming

\[ p \rightarrow q \]
\[ p \]
\[ \therefore q \]

Example: If \( n \geq 5 \), then \( n! \) is divisible by 10.

\[ n = 7 \]
\[ \therefore 7! \text{ is divisible by } 10. \]

**Modus Tollens**: means method of denying

\[ p \rightarrow q \]
\[ \sim q \]
\[ \therefore \sim p \]

Example: If \( n \) is odd, then \( n^2 \) is odd.

\[ n^2 \text{ is even.} \]
\[ \therefore n \text{ is even.} \]
More valid forms

Conjunctive simplification:

\[ p \land q \]

\[ \Rightarrow p \]

Example: The function \( f \) is 1-to-1 and continuous.

\[ \Rightarrow \text{The function } f \text{ is 1-to-1.} \]

Disjunctive addition:

\[ p \]

\[ \Rightarrow p \lor q \]

Example: The function \( f \) is increasing.

\[ \Rightarrow \text{The function } f \text{ is increasing or differentiable.} \]
More valid forms

Conjunctive addition:  
\[ p \land q \]  
\[ \therefore p \lor q \]

Example:  
\[ n \text{ is an integer, } n \text{ is positive.} \]
\[ \therefore n \text{ is a positive integer.} \]

Disjunctive syllogism:  
\[ p \lor q \]  
\[ \therefore p \land q \]

Example:  
The graph of this equation may be a circle or an ellipse.

The graph of this equation cannot be a circle.

\[ \therefore \text{The graph must be an (true) ellipse.} \]
Hypothetical syllogism:
	\[ p \supset q \]
	\[ q \supset r \]
\[ \therefore p \supset r \]

Proof by cases:
	\[ p \lor q \]
	\[ p \supset r \]
	\[ q \supset r \]
\[ \therefore r \]

Rule of contradiction:

Example:

\[ n \] is either odd or even.
If \( n \) is odd, then \( n(n-1) \) is even.
If \( n \) is even, then \( n(n-1) \) is even.
Therefore \( n(n-1) \) is always even.

Example:

If \( \sqrt{2} \) is not irrational, then there exists whole numbers \( a, b \) that are relatively prime and are both even.

Therefore \( \sqrt{2} \) must be irrational.
A valid argument with a false conclusion.

The following argument is valid by modus ponens, but since its hypothesis is false, so is its conclusion.

If $p$ is prime, then $2^p - 1$ is also prime (False).
11 is prime (True).
Therefore $2^{11} - 1$ is prime (False).

Actually, $2^{11} - 1 = 2047 = 23 \times 89$ is not prime.

Note: Any prime of the form $2^p - 1$ is called a Mersenne prime, the largest one up to date is $2^{6972593} - 1$ (discovered on 6-1-99)