1. Functions of a Complex Variable

Let $E$ be a set of complex numbers. A function $f$ defined on $E$ is a rule that assigns to each $z$ in $E$ a complex number $w$. The number $w$ is called the value of $f$ at $z$ and is denoted by $f(z)$; that is, $w = f(z)$. The set $E$ is called the domain of definition of $f$, written $D(f)$, $D(f) = \{ z \in \mathbb{C} : f(z) \text{ defined} \}$.

**Example 1**

Determine the domain definition of $f(z) = \frac{z + 1}{z^2 + z + 1}$ and $g(z) = z^2 + z + 1$

**Solution:**

\[
D(f) = \{ z \in \mathbb{C} : f(z) \text{ defined} \} = \{ z \in \mathbb{C} : z^2 + z + 1 \neq 0 \} = \{ z \in \mathbb{C} : z \neq -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3} \} \\
D(g) = \{ z \in \mathbb{C} : g(z) \text{ defined} \} = \mathbb{C}
\]

2. Limits

Let a function $f$ be defined at all points $z$ in some deleted neighborhood of $z_0$. The statement that the limit of $f(z)$ as $z$ approaches $z_0$ is a number $w_0$, or

\[
\lim_{z \to z_0} f(z) = w_0
\]

(2)

means that the point $w = f(z)$ can be made arbitrarily close to $w_0$ if we choose the point $z$ close enough to $z_0$ but distinct from it. Equation (2) means that, for each positive number $\varepsilon$, there is a positive number $\delta$ such that

\[
|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.
\]

(3)
Example 1

Show that \(\lim_{z \to i} z^2 - 1 = 2\)

Solution:

From the properties of moduli, we have

\[
\left| (z^2 - 1) - (-2) \right| = |z^2 + 1| = |z - i||z + i|.
\]

Observe that when \(\forall z \in \mathbb{C}\) is in the region \(|z - i| < 1\),

\[
|z + i| = |z - i + 2i| \leq |z - i| + |2i| < 1 + 2 = 3.
\]

Hence, for \(\forall z \in \mathbb{C}\) such that \(|z - i| < 1\),

\[
\left| (z^2 - 1) - (-2) \right| = |z^2 + 1| = |z - i||z + i| < 3|z - i|.
\]

For any positive number \(\varepsilon\), get \(\delta = \min\left\{ \frac{\varepsilon}{3}, 1 \right\}\) such that \(0 < |z - i| < \delta\),

\[
\left| (z^2 - 1) - (-2) \right| < 3|z - i| < 3 \left( \frac{\varepsilon}{3} \right) = \varepsilon.\]

Note that when a limit function \(f(z)\) exist at a point \(z_0\), it is unique.

Example 2

If \(f(z) = \frac{\overline{z}}{z}\), then \(\lim_{z \to 0} f(z)\) does not exist.

When \(z = (x, 0)\) is a nonzero point on the real axis,

\[
f(z) = \frac{x - i0}{x + i0} = 1
\]

and when \(z = (0, y)\) is a nonzero point on the imaginary axis,
Thus, by letting \( z \) approach the origin along real axis, we would find that the desired limit is 1. An approach along imaginary axis would, on the other hand, yield the limit -1. Since the limit is unique, we must conclude that \( \lim_{z \to 0} \frac{\overline{z}}{z} \) does not exist.

Since limits of the latter type are studied in calculus, we use their definition and properties freely.

**Theorem 1**

Suppose that

\[
f(z) = u(x, y) + iv(x, y), \quad z_0 = x_0 + iy_0, \quad \text{and} \quad w_0 = u_0 + iv_0.\]

Then

\[
\lim_{z \to z_0} f(z) = w_0 \quad (4)
\]

if and only if

\[
\lim_{(x,y) \to (u_0,v_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (u_0,v_0)} v(x,y) = v_0 \quad (5)
\]

**Example**

Find \( \lim_{z \to 1+i} \left( z^2 + \frac{1}{z} \right) \).

Observe that

\[
z^2 + \frac{1}{z} = (x+i) + \frac{1}{x+iy} = (x^2 - y^2) + \frac{x}{x^2 + y^2} + i \left( 2xy - \frac{y}{x^2 + y^2} \right).
\]

We have \( u(x,y) = (x^2 - y^2) + \frac{x}{x^2 + y^2} \) and \( v(x,y) = 2xy - \frac{y}{x^2 + y^2} \). From Theorem 1,

\[
\lim_{(x,y) \to (1,1)} \left( \frac{x^2 - y^2}{x^2 + y^2} + \frac{x}{x^2 + y^2} \right) = \frac{1}{2} \quad \text{and} \quad \lim_{(x,y) \to (1,1)} \left( \frac{2xy - y}{x^2 + y^2} \right) = \frac{3}{2},
\]

thus

\[
\lim_{z \to 1+i} \left( z^2 + \frac{1}{z} \right) = \frac{1}{2} + \frac{3}{2}i.
\]

**Theorem 2**

If \( \lim_{z \to z_0} f(z), \lim_{z \to z_0} g(z) \) exist and \( c \in \mathbb{C} \), then
(1) \[ \lim_{z \to z_0} (f(z) + g(z)) \text{ exist and } \lim_{z \to z_0} (f(z) + g(z)) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) \]

(2) \[ \lim_{z \to z_0} (cf(z)) \text{ exist and } \lim_{z \to z_0} (cf(z)) = c \lim_{z \to z_0} f(z) \]

(3) \[ \lim_{z \to z_0} (f(z)g(z)) \text{ exist and } \lim_{z \to z_0} (f(z)g(z)) = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) \]

(4) \[ \lim_{z \to z_0} \left( \frac{f(z)}{g(z)} \right) \text{ exist and } \lim_{z \to z_0} \left( \frac{f(z)}{g(z)} \right) = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}, \text{ whenever } \lim_{z \to z_0} g(z) \neq 0 \]

Limits Involving the Point at Infinity

We have three points about limits that is involving the point at infinity:

\[ \lim_{z \to \infty} f(z) = \infty \text{ if and only if } \lim_{z \to 0} \frac{1}{f(z)} = 0. \quad (6) \]

\[ \lim_{z \to \infty} f(z) = w_0 \text{ if and only if } \lim_{z \to 0} \frac{1}{f(z)} = w_0. \quad (7) \]

\[ \lim_{z \to \infty} f(z) = \infty \text{ if and only if } \lim_{z \to 0} \frac{1}{f(z)} = 0. \quad (8) \]

Example 1

Observe that \( \lim_{z \to \infty} \frac{i+3}{z+i} = \infty \) since \( \lim_{z \to \infty} \frac{z+i}{i+3} = 0. \)

Example 2

Observe that \( \lim_{z \to \infty} \frac{3z-i}{z+2} = 3 \) since \( \lim_{z \to 0} \frac{\frac{1}{z} - i}{\frac{1}{z} + 2} = \lim_{z \to 0} \frac{3-i}{1+2z} = 3. \)

Example 3

Observe that \( \lim_{z \to \infty} \frac{3z^4-i}{z^3+2} = \infty \) since \( \lim_{z \to 0} \frac{\frac{1}{z^3} + 2}{\frac{1}{z^3} - i} = \lim_{z \to 0} \frac{z + 2z^4}{3 - iz^4} = 0. \)

Exercises

1. For each of the functions below, describe the domain of definition that is understood

   (a) \( f(z) = \frac{1}{z^2 + 4} \) 
   (c) \( f(z) = \cos(z^2 - i) \)
(b) \( f(z) = \frac{\overline{z} + 2i}{z + \overline{z}} \)

2. Write the function \( f(z) = z^3 + 2z - i \) in the form \( f(z) = u(x, y) + iv(x, y) \).

3. Let \( z_0, c \) denote complex constant. Use definition (3) to prove that
   
   (a) \( \lim_{z \to z_0} c = c \)
   
   (b) \( \lim_{z \to 1 + i} (x + 2iy) = 1 - 2i \)

4. Let \( f(z) = \frac{z^2}{|z|^2} \)
   
   a. Find \( \lim_{z \to 0} f(z) \) along the line \( y = x \)
   
   b. Find \( \lim_{z \to 0} f(z) \) along the line \( y = 2x \)
   
   c. Find \( \lim_{z \to 0} f(z) \) along the parabola \( y = x^2 \)
   
   d. What can you conclude about the limit of \( f(z) \) along \( z \to 0 \)

5. Using (6), (7) and (8) of limits, show that
   
   (a) \( \lim_{z \to \infty} \frac{z^4 - z^3 + 2z}{(z+1)^4} = 1 \)
   
   (b) \( \lim_{z \to 2i} \frac{z}{(z-2i)^2} = \infty \)
   
   (c) \( \lim_{z \to \infty} \frac{z^2 - 1}{z + 1} = \infty \)

6. Find the value of limits below
   
   (a) \( \lim_{z \to 1 + 2i} z^2 + 2z - 1 \)
   
   (b) \( \lim_{z \to 2i} z^2 + 2z - 1 \)
   
   (c) \( \lim_{z \to (1+i)(3i)} \frac{z^3 + 8}{z^4 + 4z^2 + 16} \)
   
   (d) \( \lim_{z \to \infty} \frac{z^4 - 1}{z - i} \)
   
   (e) \( \lim_{z \to \infty} \frac{z^2 + 4z + 2}{z + 1} \)
   
   (f) \( \lim_{z \to 1+i} \frac{z^2 + z - 1 - 3i}{z^2 - 2z + 2} \)