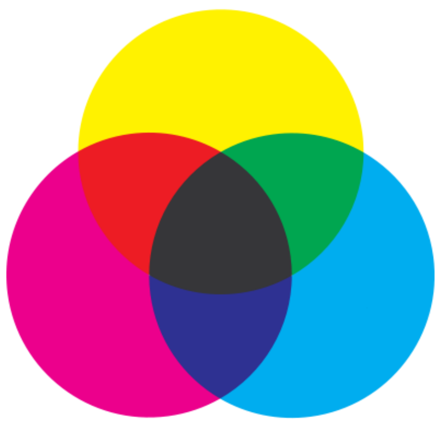
HANDOUT

LOGIC AND SET THEORY



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Subject : Logic and Set Theory
Topic : Sentence and Statement
Week : 1

A **sentence** is a set of words that is arranged grammatically and has meaning.

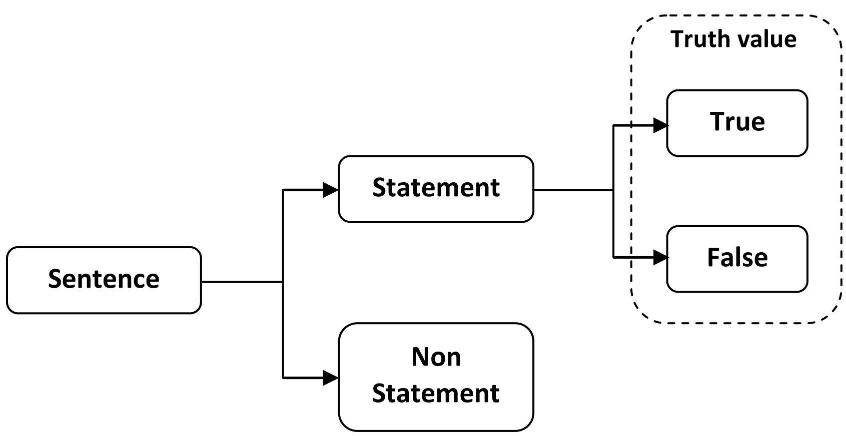
A **statement** is a sentence that is either true or false. Statement is typically a declarative sentence (i.e. a sentence that makes a declaration and consists of a subject and a predicate) or a sentence component that could stand as a declarative sentence.

Example of statement:

1. $\sin 90^o = 1$
2. 25% of a hundred is 25
3. $\cos 180^0 = 0$

Truth and falsity are called the two possible **truth values** of a statement. Thus, the truth value of the first two statements is true and the truth value of the last sentence is false.

The relation between sentence and statement is described in the following diagram:



Questions, proposals, suggestions, commands, and exclamations usually cannot be classified as statements.

Determine the truth value of the following sentences. Explain your reason. For what condition the false statements become true statements?

1. A square is a parallelogram.
2. An equilateral triangle is an isosceles triangle.
3. The sum of the interior angles of a triangle is 180^o.
4. 8+3=1
5. The altitudes of an equilateral triangle intersect orthogonally.
6. All altitudes of an isosceles triangle are also its bisector line.
7. $\sin 120^o = \sin 240^o$
8. $2 \sin 60^o = \sin 120^o$
9. 6+8=14
10. 8:2=12:3

Are the following sentences also statements?

Does a circle have diagonal(s)?

You should multiply the variables with 5

You are not allowed to divide the variables by zero

Can you draw the altitudes of an obtuse triangle?

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Subject : Logic and Set Theory
Topic : Logical Connective and Truth Table
Sub topics : negation, disjunction, conjunction
Week : 2

I. **Connective** or **logical connective**

Connective or **logical connective** is a syntactic operation on statements, or the symbol for such an operation, which corresponds to a logical operation on the logical values of those statements. The basic connectives are "NOT" (symbolized by ~),"OR" (symbolized by ∨), "AND" (symbolized by ∧), "conditional" ("if-then") or implication (symbolized by →), and "biconditional" or "if-and-only-if" (symbolized by ↔).

II. Truth table

A **truth table** is a mathematical table used in logic (specifically in classic logic) to determine the functional values of logical expressions on each of their functional arguments. In particular, truth tables can be used to tell whether a propositional expression is logically valid, that is true for all the input values.

The logical connectives and their truth table will be described as follows:

A. NEGATION

Negation is the truth function that makes truth to falsity and *vice versa*.

Symbol: "NOT" ; ~

Properties:

1. Double negation

   Double negation is the negation of the negation of a proposition p that is logically equivalent to the p itself.

$\sim(\sim p) \equiv p$

2. Self dual: $f(a_1, a_2, \dots, a_n) = \sim f(\sim a_1, \sim a_2, \dots, \sim a_n)$ for $a_1, a_2, \dots, a_n \in \{0,1\}$
   Self dual works in Boolean algebra.

Truth table of negation:

p	**~ p**
True (T)	False (F)
F	T

What is the negation of the following statements?

- A triangle which one of its interior angle is 90^o is a right triangle
- All prime numbers are odd numbers
- Some triangles have 200^o of the sum of the interior angles

B. DISJUNCTION

Disjunction is a logical operator that results in true whenever *one or more* of its operands are true.

Symbol: "OR" ; $\vee$

The properties apply to disjunction are:

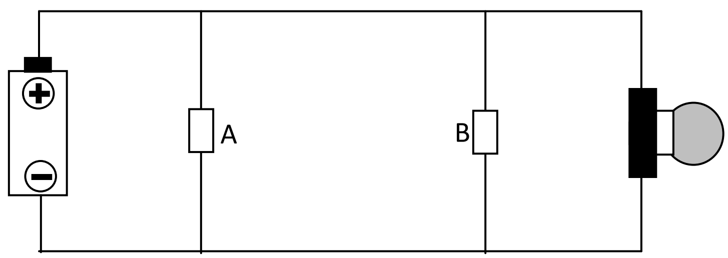
1. Associative
   $(p \vee q) \vee r \equiv p \vee (q \vee r)$
2. Commutative
   $p \vee q \equiv q \vee p$
3. Idempotent
   $p \vee p \equiv p$
4. Truth-preserving
   The interpretation under which at least one of the variables is assigned a truth value of 'true' will result a truth value 'true'.
5. Falsehood-preserving
   The interpretation under which all variables are assigned a truth value of 'false' will produce a truth value 'false'.

Truth table of a disjunction:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Disjunction and electrical system:

A disjunction is similar to the system of parallel connection in electrical circuit.



If both switch A and switch B are switched on, then the lamp is on.

If switch A is switched off and switch B is switched on, then the lamp is still on.

If switch A is switched on and switch B is switched off, then the lamp is still on.

If both switch A and switch B are switched off, then the lamp will be off.

So the lamp will be on if at least one of the switches is switched on and the lamp will be off when all switches are switched off.

The condition for a parallel connection of electrical system:

S_1 (Switch 1)	S_2 (Switch 2)	Lamp
ON	ON	ON
ON	OFF	ON
OFF	ON	ON
OFF	OFF	OFF

C. CONJUNCTION

Conjunction is a two-place logical connective that has the value *true* if both of its operands are true, otherwise a value of *false*.

Symbol: "AND" ; ∧

The properties apply to conjunction are:

1. Associative

   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

2. Commutative

   $p \wedge q \equiv q \wedge p$

3. Distributive

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

4. Idempotent

   $p \ \wedge p \ \equiv p$

5. Truth-preserving

   The interpretation under which all variables are assigned a truth value of 'true' will produce a truth value 'true'.

6. Falsehood-preserving

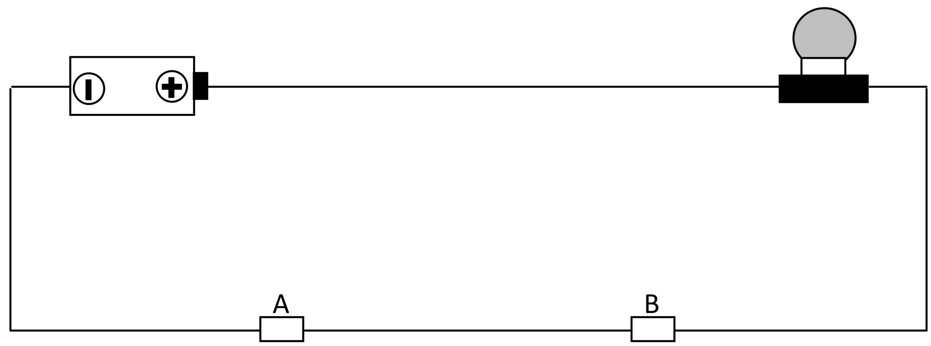
   The interpretation under which at least one of the variables is assigned a truth value of 'false' will result a truth value 'false'.

Truth table of a conjunction:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction and electrical system:

A conjunction is similar to the system of serial connection in electrical circuit.



If both switch A and switch B are switched on, then the lamp is on.

If switch A is switched off and switch B is switched on, then the lamp will be off.

If switch A is switched on and switch B is switched off, then the lamp will be off.

If both switch A and switch B are switched off, then the lamp will be off.

So the lamp will be on if at least one of the switches is switched on and the lamp will be off when all switches are switched off.

The condition for a serial connection of electrical system:

S_1 (Switch 1)	S_2 (Switch 2)	Lamp
ON	ON	ON
ON	OFF	OFF
OFF	ON	OFF
OFF	OFF	OFF

Note:

The truth value of a conjunction and a disjunction is determined by the truth values of its operands, without considering the relation (of sentence) between the operands.

Example:

One is a prime number or a triangle has three sides.

The truth value of this disjunction is true although there is no relation between one as a prime number and the three sides of a triangle. The truth value of the disjunction is determined by the truth value of 1 as a prime number and the three sides of a triangle.

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Subject : Logic and Set Theory
Topic : Logical Connective and Truth Table
Sub topics : implication, logical equivalence/biconditional and the negation of logical connectives
Week : 3

D. IMPLICATION (CONDITIONAL)

Implication is a logical operator connecting two statements to assert if *p* then *q* or *q* only if *p*; where *p* is an *antecedent* and *q* is a *consequent*.

An implication produces a value of *false* just in the singular case the first operand (i.e. *antecedent*) is true and the second operand (i.e. *consequent*) is false.

Symbol: "IF-THEN" ; → ; ⇒

Truth table of an implication:

p	q	p ⇒ q
T	T	T
T	F	F
F	T	T
F	F	T

Now, look at the following truth table:

p	q	~p	~p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

From the tables, it can be observed that the truth values of $p \Rightarrow q$ are same with the truth values of $\sim p \vee q$. Therefore, it can be concluded that $(p \Rightarrow q) \equiv (\sim \mathbf{p} \vee \mathbf{q})$.

There are three operations work in implication (conditional)

1. **Inverse**

   A statement is the **inverse** of the other statement when its antecedent is the negated antecedent of the other statement and its consequent is the negated consequent of the other statement.

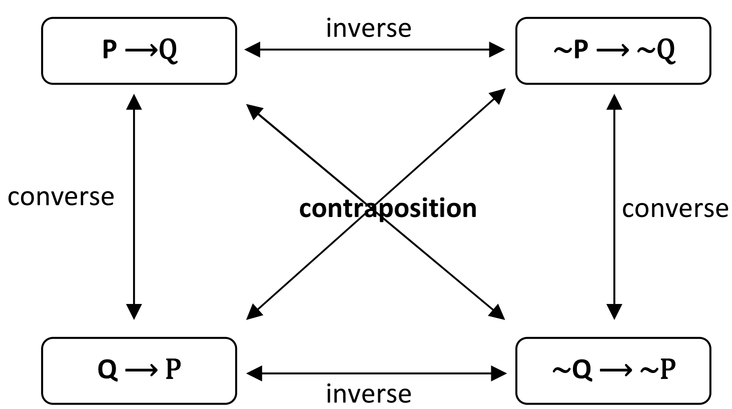
2. **Converse**

   A statement is the **converse** of the other statement when its antecedent is the consequent of the other statement and its consequent is the antecedent of the other statement.

3. **Contraposition**

   A statement is the **contrapositive** of the other statement when its antecedent is the negated consequent of the other statement and its consequent is the negated antecedent of the other statement.

The relation among inverse, converse and contraposition of a statement is described by the following diagram:



To think about:

1. Is “if a shape is a square, then it has two pairs of parallel sides” equivalent with “if a shape has two pairs of parallel sides, then it is a square”? Explain your reasoning!
2. Make a truth table for an implication and its inverse, converse and contraposition. For which operation is implication equivalent to?

E. LOGICAL EQUIVALENCE (BICONDITIONAL)

Biconditional is a logical operator connecting two statements to assert p if and only if q. Biconditional will have a truth value TRUE if both operands have same truth value.

Symbol: "IF AND ONLY IF" ; ↔ ; ⇔

Biconditional can be said as a two-directional implication. Therefore, $\boldsymbol{p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)}$ or $\boldsymbol{p \Leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)}$.

Truth table of a biconditional:

p	q	p ⇔ q
T	T	T
T	F	F
F	T	F
F	F	T

F. NEGATION OF LOGICAL CONNECTIVES

If the negation of a single preposition will result a single negated preposition, **the negation of logical connectives with result negated prepositions (operands) and negated logical operation.**

1. Negation of disjunction and conjunction

   Look at the following table:

p	q	~p	~q	p ∨ q	p ∧ q	~(p ∨ q)	~(p ∧ q)	~p ∨ ~q	~p ∧ ~q
T	T	F	F	T	T	F	F	F	F
T	F	F	T	T	F	F	T	T	F
F	T	T	F	T	F	F	T	T	F
F	F	T	T	F	F	T	T	T	T

   From the table it is obvious that the truth values of **~(p ∨ q)** is same with the truth values of **~p ∧ ~q** and the truth values of **~(p ∧ q)** is same with the truth values of **~p ∨~q**. Therefore, it can be concluded that:

   - The negation of a disjunction is the conjunction of its negated-operands

     $\sim (\boldsymbol{p} \vee \boldsymbol{q}) \equiv \sim \mathbf{p} \wedge \sim \mathbf{q}$

- The negation of a conjunction is the disjunction of its negated-operands.

  $\sim(\boldsymbol{p} \wedge \boldsymbol{q}) \equiv \sim \mathbf{p} \vee \sim \mathbf{q}$

2. Negation of implication and biconditional

   Because $(\boldsymbol{p} \Rightarrow \boldsymbol{q}) \equiv (\sim \mathbf{p} \vee \mathbf{q})$ and $\boldsymbol{p} \Leftrightarrow \boldsymbol{q} \equiv (\sim\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim\boldsymbol{q} \vee \boldsymbol{p})$, so the negation of implication and biconditional can be determined by using the rule of the negation of disjunction and conjunction (without using truth table), namely:

   - $(\boldsymbol{p} \Rightarrow \boldsymbol{q}) \equiv (\sim \mathbf{p} \vee \mathbf{q})$ therefore $\sim(\boldsymbol{p} \Rightarrow \boldsymbol{q}) \equiv \sim(\sim \mathbf{p} \vee \mathbf{q}) \equiv (\boldsymbol{p} \wedge \sim\boldsymbol{q})$
   - $(\boldsymbol{p} \Leftrightarrow \boldsymbol{q}) \equiv (\sim\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim\boldsymbol{q} \vee \boldsymbol{p})$ therefore

     $\sim(\boldsymbol{p} \Leftrightarrow \boldsymbol{q}) \equiv \sim\big((\sim\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim\boldsymbol{q} \vee \boldsymbol{p})\big) \equiv (\boldsymbol{p} \wedge \sim\boldsymbol{q}) \vee (\boldsymbol{q} \wedge \sim\boldsymbol{p})$

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Subject : Logic and Set Theory

Topic : Tautology, Contradiction and Contingency

Week : 4

A. TAUTOLOGY

Tautology is a propositional formula that is always true under any possible valuation of its propositional variables.

Symbol: T_o

Example of tautology:

- $p \vee \sim p$
- $(p \vee q) \Leftrightarrow (q \vee p)$
- $(p \wedge q) \vee \sim p \vee \sim q$

The problem of determining whether a formula is a tautology is fundamental in propositional logic. An algorithmic method of verifying that every valuation causes this sentence to be true is to make a truth table that includes every possible valuation.

There are three main properties of tautology, namely:

1. Identity

   Tautology is identity for conjunction because the conjunction of a proposition and tautology will result the truth value of the proposition.

   $p \wedge T_o \equiv p$

2. Domination

   Tautology is domination for disjunction because the disjunction of a proposition and tautology will always result a true truth value (i.e. the truth value of tautology)

   $p \vee T_o \equiv T_o$

3. Inverse

   For disjunction, a preposition and its negation are inverse each other because ther disjunction will result tautology.

   $p \lor {\sim}p \equiv T_o$

B. CONTRADICTION

Contradiction is the negation of tautology; therefore contradiction can be defined as a propositional formula that is always false under any possible valuation of its propositional variables.

Symbol: ${\sim}T_o$

Example of contradiction:

- $p \land {\sim} p$
- $({\sim}p \land {\sim}q) \Leftrightarrow (q \lor p)$
- $(p \lor q) \land {\sim}p \land {\sim}q$

As well as tautology, determining whether a formula is a contradiction is fundamental problem in propositional logic. An algorithmic method of verifying that every valuation causes this sentence to be false is to make a truth table that includes every possible valuation.

There are three main properties of contradiction, namely:

1. Identity

   Contradiction is identity for disjunction because the disjunction of a proposition and contradiction will result the truth value of the proposition.

   $p \lor {\sim}T_o \equiv p$

2. Domination

   Contradiction is domination for conjunction because the conjunction of a proposition and contradiction will always result a false truth value (i.e. the truth value of contradiction)

   $p \land {\sim}T_o \equiv {\sim}T_o$

3. Inverse

   For conjunction, a preposition and its negation are inverse each other because their conjunction will result contradiction.

   $p \wedge \sim p \equiv \sim T_o$

C. CONTINGENCY

Contingency is the status of propositions that are not necessarily true or necessarily false. A formula that is neither a tautology nor a contradiction is said to be logically contingent. Such a formula can be made either true or false based on the values assigned to its propositional variables.

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Subject : Logic and Set Theory
Topic : Deriving conclusion
Week : 5

In logic, an argument is a group of statements in which some of them (called as premises) provides support for reasons to believe another statement (called as conclusion). The premises are the statements that set forth the reasons or evidence, and the conclusion is the statement that the evidence is claimed to support or imply. In other words, the conclusion is the statement that is claimed to follow from the premises. The statements that make up an argument are divided into one or more premises and one and only one conclusion. Therefore, one of the most important tasks in logic is how to derive a conclusion from given premises. There some methods used in logic to derive conclusion, namely:

1. Syllogism

A categorical syllogism consists of three parts, namely:

1. The major premise,
2. The minor premise, and
3. The conclusion.

Each of the premises has **one term in common** with the conclusion: in a major premise, this is the *major term* (*i.e.,* the predicate of the conclusion); in a minor premise, it is the *minor term* (i.e. the subject of the conclusion).

Example:

Premise 1 : All kinds of triangle have three altitudes
Premise 2 : Right triangle is a kind of triangle
Conclusion : Right triangle has three altitudes

From the given example, we can find that “have/has three altitudes” is the *major term* that is contained in premise 1 (major premise) and the conclusion. The *minor term* is “right triangle” which is contained in premise 2 (minor premise) and the conclusion.

Syllogism can be expressed in the following way:

Premise 1 : $P \Rightarrow Q$

Premise 2 : Q $\Rightarrow R$

Conclusion : P $\Rightarrow R$

2. Modus ponendo ponens or the law of detachment

   In simple words, modus ponendo ponen is called as **affirming the antecedent**.

   In modus ponendo ponens, the argument has two premises. The first premise is an implication or conditional claim (e.g. if P then Q). The second premise is merely the antecedent of the first premise (i.e. P) which is true. From these two premises it can be logically concluded that the consequent of the conditional claim (i.e. Q) must be true as well.

   Example:

   Premise 1 : If a shape has two pairs of parallel sides, then it is a parallelogram

   Premise 2 : A shape has two pairs of parallel sides

   Conclusion : It is a parallelogram

   Modus ponendo ponens can be expressed in the following way:

   Premise 1 : $P \Rightarrow Q$

   Premise 2 : P

   Conclusion : Q

3. Modus tollendo tollens

   In contrary to modus ponendo ponen which emphasis on the antecedent, modus tollendo tollens focuses on the consequent. Modus tollendo tollens is also called as modus which is denying the consequent.

In modus tollendo tollens, the argument has also two premises. The first premise is an implication or conditional claim (e.g. if P then Q). The second premise is that the consequent of the first premise is false. From these two premises it can be logically concluded that the antecedent of the conditional claim must be false.

Example:

Premise 1 : If a shape has two pairs of parallel sides, then it is a parallelogram

Premise 2 : A shape is not a parallelogram

Conclusion : It does not have two pairs of parallel sides

Modus tollendo tollens can be expressed in the following way:

Premise 1 : $P \Rightarrow Q$

Premise 2 : $\sim Q$

Conclusion : P

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Subject : Logic and Set Theory
Topic : Quantification
Week : 6

In this topic, we are going to discuss: open sentence and quantification (i.e. universal quantifier and existential quantifier).

A. Open sentence

An open sentence is a sentence which contains variable(s) and this sentence will become a statement if the variable(s) is substituted by constant(s). The truth value of an open sentence is meaningless until its variables are replaced with specific numbers, at which point the truth value can usually be determined.

Example:

$x + 2 < 3$ is an open sentence because it contains variable x and we cannot determine its truth value.

If we substitute the variable x with any number, then this sentence will be a statement. For example if we substitute the x with 3, then the open sentence will be a false statement, but if we substitute the x with 0, then the open sentence will be a true statement.

B. Universal Quantifier and Existential Quantifier

We have discussed that we can change an open sentence into a statement by substituting the variable(s) with constant(s). Another way to change an open sentence into a statement is by using quantification. Quantification is the binding of a variable ranging over a domain of discourse. The operator used in quantification is called a quantifier. There are two kinds of quantifiers, namely universal quantifier and existential quantifier. Universal quantifier is expressed by "for all" and symbolized by "∀". Existential quantifier is expressed by "there exists" and symbolized by "∃".

Example:

$x + 2 < 3$ is an open sentence

- If we add existential quantifier to this sentence then it will become: $(\exists x)(x + 2 < 3)$ that can be read as "there exist x that makes $x + 2 < 3$. It is obvious that the new sentence is a statement because we can determine its truth value that is true.
- If we add universal quantifier to this sentence then it will become: $(\forall x)(x + 2 < 3)$ that can be read as "for all x it will be true that $x + 2 < 3$". It is obvious that the new sentence is a statement because we can determine its truth value that is false (because not for all x satisfy $x + 2 < 3$, such as $x = 5$).

Universal quantifier and existential quantifier is a negation for each other.

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Subject : Logic and Set Theory
Topic : Set
Sub topics : definition of a set, relation and operation on a set, and properties of a set
Week : 8

A. Definition of a set

A set is a collection of distinct objects either concrete or abstract which has a well-defined criterion for its members (later, the member of a set is called as element). Well-defined criterion for its elements means that for any object, it can be easily defined whether this object is an element of the set or not.

Example:

The set of odd numbers.

The elements of the set are defined by $2n - 1$ for all $n \in$ integer numbers. From this well-defined criterion for element, it can easily be concluded that 2 is not an element of this set because for 2 does not satisfy $2n - 1$.

A set is symbolized by a capital letter. For example O is a symbol for the set of odd numbers.

A set which does not have any element is called *empty set* or *null set*. *Empty set* is symbolized by $\emptyset$ or $\{\ \ \}$.

$\{x \in R | x^2 < -1\}$ is an example of an *empty set* because there is no real number which its square results a negative number.

B. Relation of Sets

The relations of sets are described as follows:

1. Subset

   For two different sets A and B, A is a subset of B (noted as "$A \subset B$") if all elements of A are also elements of B.

Example:

$A = \{4,16\}$ and $B = \{1,4,9,16,15\}$

It can easily be seen that all elements of A are also the elements of B, therefore $A \subset B$.

From the definition of subset, it can be derived the following propositions:

- *Empty set* is the sub set of every set.
- If $A \subset B$ and $B \subset C$ then $A \subset C$

2. Equality

   For two sets A and B, A will be equal to B (noted as "$A = B$") if $A \subset B$ and $B \subset A$.

   Example:

   $A = \{1,3,5\}$ and $B = \{5,3,1\}$

   Because $A \subset B$ and $B \subset A$, therefore $A = B$.

3. Proper subset

   For two sets A and B, A is called as a proper subset of B if all elements of A are elements B but at least one of the elements of B are not the element of A.

4. Equivalent

   Two sets are equivalent if their elements can be put into one-to-one correspondence with each other.

C. Operations on set

There are some operations apply on set, namely:

1. Union

   The union of two sets *A* and *B* is the set that contains all elements of *A* and *B*.

   The union of two sets *A* and *B* is symbolized as $A \cup B = \{x | x \in A \ or \ x \in B\}$

2. Intersection

   The intersection of two sets *A* and *B* is the set that contains elements of *A* which also belong to *B* (or equivalently, all elements of *B* that also belong to *A*), but no other elements.

The intersection of two sets A and B can be symbolized as $A \cap B = \{x | x \in A \text{ and } x \in B\}$

3. Disjoint

   Two sets *A* and *B* is said to be disjoint if the intersection of sets *A* and *B* is empty or they have no elements in common.

   This condition is symbolized as $A \cap B = \emptyset$

4. Subtraction of set

   The subtraction of set *A* by set *B* is symbolized as $A \setminus B = \{x \in A | x \notin B\}$.

   For given set X and set *A*. If $A \subset X$, then there is an A^c (called complement of *A*) as the result of $X \setminus A$.

   Example:

   $X = \{1,2,3,4,5\}$ and $A = \{1,3,5\}$ so $A^c = \{2,4\}$

D. Properties of a set

For arbitrary sets *A, B* and *C*, the following properties of set apply:

1. Commutative
   - $A \cup B = B \cup A$
   - $A \cap B = B \cap A$
2. Associative
   - $(A \cup B) \cup C = A \cup (B \cup C)$
   - $(A \cup B) \cup C = A \cup (B \cup C)$
3. Distributive
   - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. Double complement

   $(A^c)^c = A$
5. De Morgan Law:
   - $(A \cup B)^c = A^c \cap B^c$
   - $(A \cap B)^c = A^c \cup B^c$

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Subject : Logic and Set Theory

Topic : Set

Sub topics : Ordered pairs, Cartesian product, power set

Week : 9

A. Ordered Pairs

An ordered pair is a collection of objects having two *entries*, such that one can always uniquely determine the object.

If the first entry is *a* and the second is *b*, the usual notation for the ordered pair is (*a*, *b*). The pair is "ordered" in that (*a*, *b*) differs from (*b*, *a*) unless *a*=*b*. Therefore, the property of ordered pairs is that $(x, y) = (x^*, y^*)$ if $x = x^*$ and $y = y^*$.

The entries of an ordered pairs can be other ordered pairs. For example for the ordered triple (a, b, c) can be defined as $\left(a, (b, c)\right)$.

B. Cartesian Product

Cartesian product of set *X* and set *Y* is a direct product of two sets (set *X* and set *Y*) that will result in a set of all ordered pairs whose all the first entries are the element of set *X* and the second entries are the element of set *Y*.

Given a family of sets $\{A_1, A_2, \dots, A_n\}$, the Cartesian product of this family is:

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) | \, a_i \in A_i \; for\ every\ i = 1,2, \dots, n\}$$

Example:

Given $A = \{1,4,9\}$ and $B = \{x, y\}$ then:

- $A \times B = \{(1, x), (1, y), (4, x), (4, y), (9, x), (9, y)\}$
- $B \times A = \{(x, 1), (x, 4), (x, 9), (y, 1), (y, 4), (y, 9)\}$

Remember that $(x, y) = (x^*, y^*)$ if $x = x^*$ and $y = y^*$, therefore $A \times B \neq B \times A$.

For arbitrary sets *A, B, C* and *D*, the following properties apply for their Cartesian product:

- $A \times B \neq B \times A$
- $A \times \emptyset = \emptyset \times A$
- $(A \times B) \times C \neq A \times (B \times C)$
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
- $(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$

C. Power Set

Given a set *A*, the power set of A is the set of all subsets of *A*.

Power set of *A* is notated as $P(A)$, therefore $P(A) = \{X | X \subset A\}$. Since *empty set* is a subset of every set, therefore the power set of any set always contains $\emptyset$.

Example:

$A = \{1,3,5\}$, therefore $P(A) = \big\{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,3,5\}\big\}$

$B = \{4\}$, therefore $P(A) = \big\{\emptyset, \{4\}\big\}$

$P(\emptyset) = \{\emptyset\}$

From the given examples, we can see that the number of elements of a power set of set is given by formula 2^n, for *n* is the number of elements of the set.

Mathematics Education Department

Faculty of Mathematics and Natural Science

Yogyakarta State University

Subject : Logic and Set Theory

Topic : Relation and Map

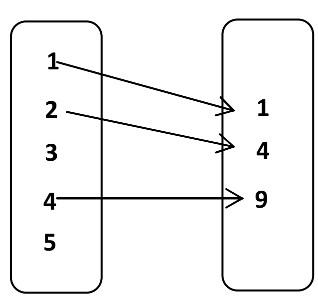
Week : 10

A. Definition

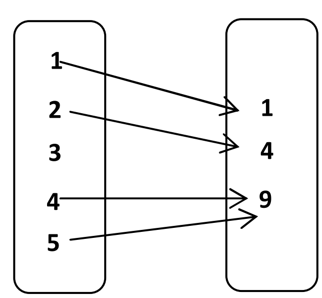
In set theory, a relation is just a relationship between sets of information. If we connect elements of a set to the elements of another set, it is called relation.

Map is also a kind of relation but it has specific rule, namely each element of the first set (later will be called as domain) must be related/connected to only one element of second set (called as codomain). In simple words, a relation is called as a map if for each element of the domain only has one "partner" in the codomain.

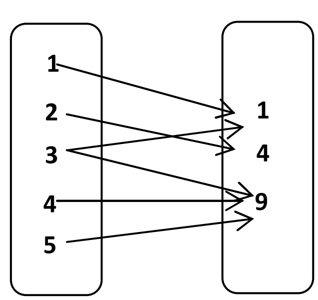
Example:



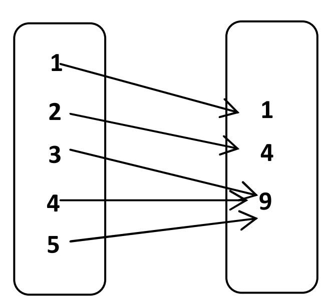
A



B



C



D

All the examples are relation, but not all of them are map. Example (A) and (B) are not map because there is an element of domain which does not have any partner in the codomain. Example (C) is also not a map because there is an element of domain which has two partners in the codomain. Example (D) is a map because all elements of domain have single partner in the codomain, although some elements of domain have same partner in the codomain.

B. Types of Mapping

There are three types of mapping, namely:

1. *Onto* Map

   A map is called an *onto* map if for all the elements of codomain has partner in the domain.

2. *One-to-one* Map

   A map is called a *one-to-one* map if there is no element of codomain which has no more than one partners in the domain.

3. *Bijective* Map

   If a map is an *onto* map and also a *one-to-one* map, then this map is called *bijective* map.

We will discuss these three types of mapping again when we discuss the type of functions, because in the set theory mapping and function are highly related.

Mathematics Education Department

Faculty of Mathematics and Natural Science

Yogyakarta State University

Subject : Logic and Set Theory

Topic : Function

Sub topics : definition of a function, types of function, inverse function

Week : 11

A. Definition

Function is closely related to mapping. However, a function is applied for number whereas a map is for general set (not only number). A function also refers to an operation that is used in a map.

A function associates a unique value to each input of a specified type.

B. Types of function

Due to the relation between function and mapping, types of function are also closely related to the types of mapping. The types kinds of functions are similar to the three types of mapping, namely:

1. *Onto* function or also called surjective function

   A function given for a map $f: X \to Y$ is called *onto* or surjective if for every y in codomain, there is an x in the domain such that $f(x) = y$.

   Example:

   $\boldsymbol{Z}$ is the set of integer numbers

   $\boldsymbol{N} \cup \{\boldsymbol{0}\}$ is the set of positive integer numbers

   Map $f: Z \to N \cup \{0\}$ with the function $f(x) = |x|$

   This function is a surjective function because for every $y \in N \cup \{0\}$ there is an $x \in Z$, namely: $x = n$ and $x = -n$.

2. *One-to-one* function or injective function

A function given for map $f: X \rightarrow Y$ is called *one-to-one* or injective if for $a, b \in X$ which $a \neq b$, then $f(a) \neq f(b)$.

Example:

Given a map $f: R \rightarrow R$ with the function is $f(x) = 2x + 1$

This function is an injective function because if there is $f(m) = f(n)$, then $m = n$.

3. *Bijective* function

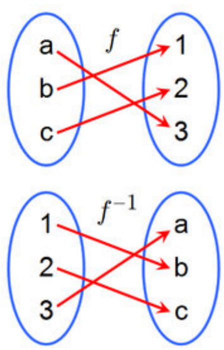
A function given for a map $f: X \rightarrow Y$ is called bijective if the function is surjective and also injective.

Example:

- For a given map $f: Z \rightarrow Z$, function is $f(x) = 2 - x$ is a bijective function
- For a given map $f: Z \rightarrow Z$, function is $f(x) = 2x$ is not a bijective function because odd numbers in the codomain do not have partner in the domain

C. Inverse function

Given a bijective function for map $f: A \rightarrow B$. The function for map $g: B \rightarrow A$ defined by $g(b) = a$ for $f(a) = b$ is, then, the inverse of function *f*. The inverse of function *f* is notated by f^{-1}.



Source of the image: www.wikipedia.org

Example:

$f(x) = 2x + 1$

$x = \frac{f(x)-1}{2}$

$f^{-1}(x) = \frac{f\left(f^{-1}(x)\right)-1}{2}$

$f^{-1}(x) = \frac{x-1}{2}$

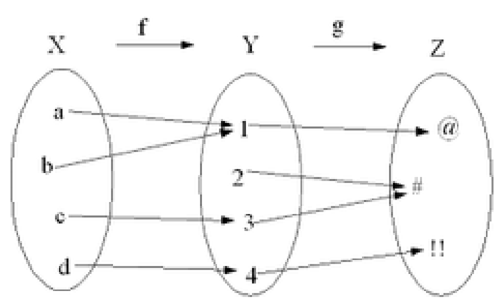
So the inverse of $f(x) = 2x + 1$ is $f^{-1}(x) = \frac{x-1}{2}$

Mathematics Education Department
Faculty of Mathematics and Natural Science
Yogyakarta State University

Subject : Logic and Set Theory
Topic : Function
Sub topics : composite function and properties of a function
Week : 12

A. Definition of composite function

Given two functions f and g for map $f: A \to B$ and $g: B \to C$. It can be generated a new function for new map which directly maps from A to C, namely $g \circ f: A \to C$. The new function is called composite function notated by $g \circ f(x) = g(f(x))$ for every $x \in A$.



Source of the image: www.wikipedia.org

Example:

Given maps $f: R \to R$ and $g: R \to R$ with $f(x) = x^2$ and $g(x) = 2x - 1$.

The possible composite function can be generated are:

- $g \circ f(x) = g(f(x))$
  $g \circ f(x) = 2(x^2) - 1$
  $g \circ f(x) = 2x^2 - 1$

- $f \circ g(x) = f(g(x))$
  $f \circ g(x) = (2x - 1)^2$

$f \; o \; g(x) = 4x^2 - 4x + 1$

B. Properties of a function

For *A* and *B* are arbitrary sets, the following properties apply for a function:

1. If $A \subset B$, then $f(A) \subseteq f(B)$
2. $f(A \cup B) = f(A) \cup f(B)$
3. $f(A \cap B) \subseteq f(A) \cap f(B)$
4. $f(A) - f(B) \subseteq f(A - B)$
5. $A \subseteq f^{-1}f(A)$
6. If $A \subset B$, then $f^{-1}(A) \subseteq f^{-1}(B)$
7. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
8. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Mathematics Education Department

Faculty of Mathematics and Natural Science

Yogyakarta State University

Subject : Logic and Set Theory

Topic : Set (advanced)

Week : 14

A. Denumerable Set and Non-denumerable Set

Based on the number of the elements, there are two kinds of set, namely finite set and infinite set. A set is called an infinite set if this set is equipotent (i.e. sets that provide bijective map/function) with its proper subset. Consequently, B is an infinite set if $A \subset B$ and A is infinite set.

There are two kinds of infinite set, namely *denumerable set* and *non-denumerable set*.

1. Denumerable set

The term denumerable means countably infinite. An infinite set is *denumerable* if it is equivalent to the set of natural numbers.

The following sets are all denumerable sets:

– The set of natural numbers

Of course, the set of natural numbers is denumerable because it is equivalent with the set of natural number (i.e. itself).

– The set of integers

The set of integers is denumerable because it is equivalent with the set of natural numbers if we give the map $f: N \rightarrow Z$ with the following functions:

$f(n) = \frac{n}{2}$, for n is an even number

$f(n) = \frac{1-n}{2}$, for n is an odd number

– The set of odd integers

The set of odd integers is denumerable because it is equivalent with the set of natural numbers if we give map $f: N \rightarrow Z$ with function $f(n) = n$ for all $n \in N$

- The set of even integers

  The set of even integers is denumerable because it is equivalent with the set of natural numbers if we give the map $f: N \to Z$ with fungsi $f(n) = 2n$ for all $n \in N$

2. **Non-denumerable set**

   Non-denumerable set is a set that is infinite and uncountable.

B. Cardinal Number

A cardinal number is a generalization of the natural numbers used to measure the cardinality or size of sets. The cardinality of a finite set is a natural number that refers to the number of elements in the set and the size of infinite set is described by transfinite cardinal numbers (i.e. numbers that are larger than all finite numbers, yet not necessarily absolutely infinite). The cardinal number of set *A* is notated by $|A|$.

Two sets have the same cardinal number if and only if there is a bijection between them. For finite sets, this agrees with the intuitive notion of size of a set.

Example:

The following pairs of sets have same cardinal number:

1. $\{1,4,9,16\}$ and $\{a, b, c, d\}$
2. $\{1,3,5,7, \ldots\}$ and $\{2,4,6,8, \ldots\}$
3. *N* and *Z*

Given arbitrary sets A and B. If there is a one-to-one mapping $f: A \to B$, then $|A| \leq |B|$. The relation "≤" has two properties, namely reflective and transitive.

- $|A| \leq |A|$ (reflective)
- If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$ (transitive)

In the case of infinite sets, it is possible for infinite sets to have different cardinalities, and in particular the set of real numbers and the set of natural numbers do not have the same cardinal number. It is also possible for a proper subset of an infinite set to have the same cardinality as the original set, something that cannot happen with proper subsets of finite sets.