Second-Order Systems

- More parameters that describe the response.
- Varying ‘a’ for first order systems simply changes the speed of the response.
- Varying the parameters of a second order response changes the speed and the form of the response.

Overdamped Response

- Consider the step response for the following system:

\[ G(s) = \frac{9}{s^2 + 9s + 9} \]

\[ \Rightarrow C_{\text{step}}(s) = \frac{9}{s(s^2 + 9s + 9)} \]

\[ = \frac{9}{s(s + 7.854)(s + 1.146)} \]

- The output has a pole at \( p_1=0 \) resulting from \( u(t) \) and \( p_2=-7.854 \) and \( p_3=-1.146 \) coming from the system itself.
(a) \[ G(s) = \frac{b}{s^2 + as + b} \]

General

(b) \[ G(s) = \frac{9}{s^2 + 9s + 9} \]

Overdamped

c(t) \[ c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t} \]

(c) \[ G(s) = \frac{9}{s^2 + 2s + 9} \]

Underdamped

c(t) \[ c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8} \sin\sqrt{8}t) = 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°) \]

(d) \[ G(s) = \frac{9}{s^2 + 9} \]

Undamped

c(t) \[ c(t) = 1 - \cos 3t \]

(e) \[ G(s) = \frac{9}{s^2 + 6s + 9} \]

Critically damped

c(t) \[ c(t) = 1 - 3te^{-3t} - e^{-3t} \]
• The input pole generates the constant forced response. The two system poles on the real axis generates the exponential response:

\[ K_1 e^{-7.854t}, K_2 e^{-1.146t} \]

• Known as overdamped response.

**Underdamped Response**

• Given

\[ C(s) = \frac{9}{s(s^2 + 2s + 9)} \]

\[ c(t) = 1 - e^{-t} \left( \cos \sqrt{8}t + \frac{\sqrt{8}}{8} \sin \sqrt{8}t \right) \]

• Poles at:

\[ s = 0 \text{ (due to input step)} \]

\[ s = -1 \pm j\sqrt{8} \text{ (natural response)} \]

  o Real part of pole matches the exponential decay frequency of the sinusoid’s amplitude.

  o Imaginary part matches frequency of the sinusoidal oscillation. Define this frequency as \( \omega_d \), i.e. damped frequency of oscillation, equals \( \sqrt{8} \) in this case.
Undamped Response

- Given

\[ C(s) = \frac{9}{s(s^2 + 9)} \]

\[ c(t) = 1 - \cos 3t \]

- Two imaginary poles at \( \pm j3 \) generate a sinusoidal natural response with frequency \( \omega = 3 \text{ rad/s} \).

- No real part implies an exponential that does not decay.
Critically Damped Response

- Given
  \[ C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s + 3)^2} \]
  \[ c(t) = 1 - 3te^{-3t} - e^{-3t} \]

- Pole at origin due to the step input and two real poles at \( p_1, p_2 = -3 \) from the system itself.

- Response consists of exponential and exponential times \( t \). Exponential parameter equals the location of the real poles.

- Responses are fastest possible without the overshoot.

Unstable

- Poles are located on the right half plane.

- Results in positive exponentials. Response goes to infinity as time increases.

- Example:
  \[ C(s) = \frac{9}{s(s - 3)} = \frac{-3}{s} + \frac{3}{s - 3} \]
  \[ c(t) = -3 + e^{3t} \]
Summary of Natural Responses

1. Overdamped responses:
   ◦ Poles: two real at \(-\sigma_1, -\sigma_2\).
     \[ c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t} \]

2. Underdamped responses:
   ◦ Poles: two complex at \(-\sigma_d \pm j \omega_d\)
     \[ c(t) = K_1 e^{-\sigma_d t} \left( K_2 \cos \omega_d t + K_3 \sin \omega_d t \right) \]

3. Undamped responses:
   ◦ Poles: 2 imaginary at \(\pm j \omega_1\)
     \[ c(t) = A \cos \omega_1 t \]

4. Critically damped responses:
   ◦ Poles: Two real at \(-\sigma_1\)
     \[ c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_2 t} \]

5. Unstable:
   ◦ Poles: Any right half plane poles, e.g. at \(\sigma_d \pm j \omega_d\).
     \[ c(t) = K_1 e^{\sigma_d t} \left( K_2 \cos \omega_d t + K_3 \sin \omega_d t \right) \]