Second-Order Systems

- More parameters that describe the response.
- Varying '*a*' for first order systems simply changes the speed of the response.
- Varying the parameters of a second order response changes the speed and the form of the response.

Overdamped Response

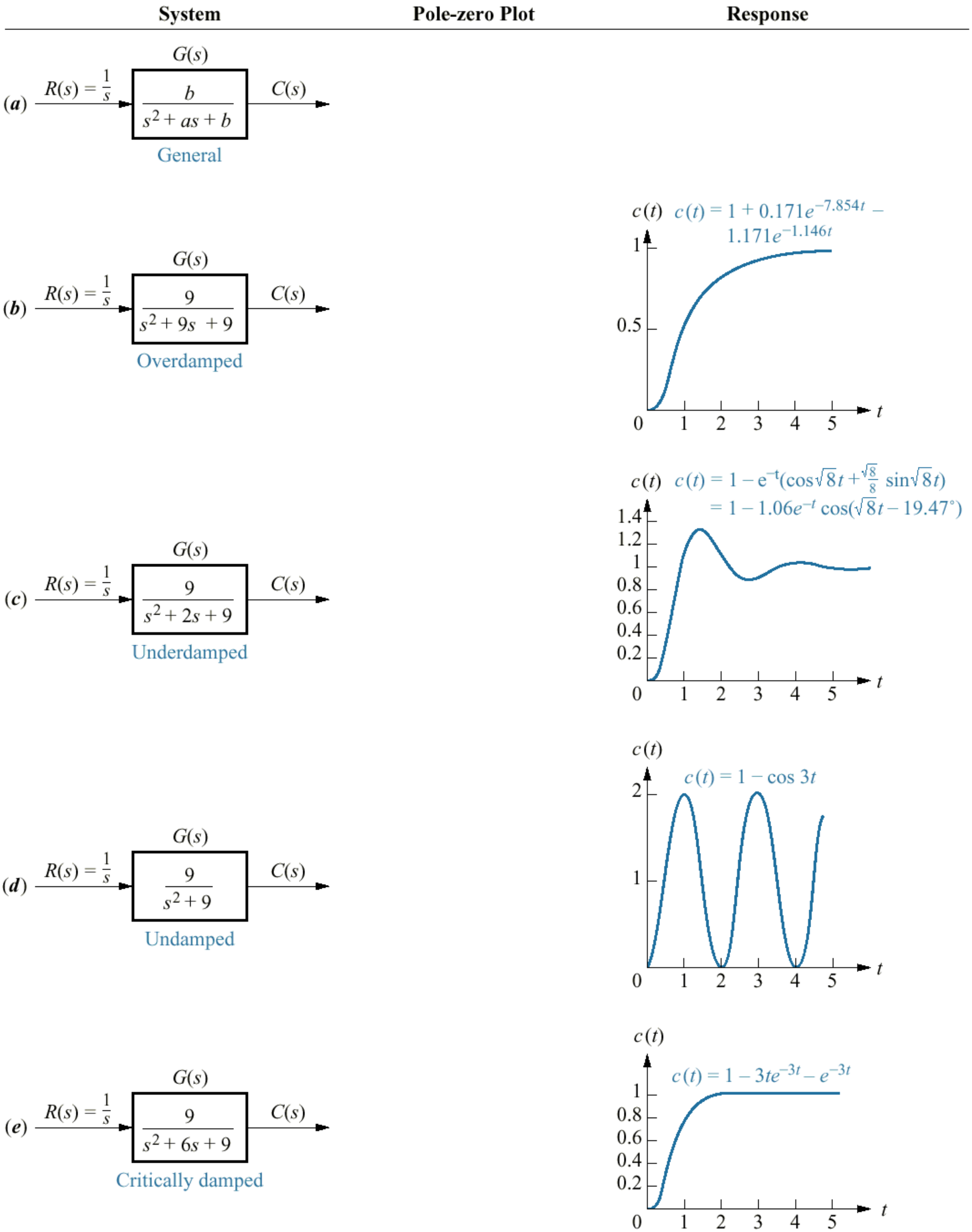
- Consider the step response for the following system:

$$G(s) = \frac{9}{s^2 + 9s + 9}$$

$$\Rightarrow C_{step}(s) = \frac{9}{s(s^2 + 9s + 9)}$$

$$= \frac{9}{s(s + 7.854)(s + 1.146)}$$

- The output has a pole at $p_1=0$ resulting from $u(t)$ and $p_2=-7.854$ and $p_3=-1.146$ coming from the system itself.


System
Pole-zero Plot
Response
(a) R(s) = 1/s
G(s)
b / (s^2 + as + b)
C(s)
General
(b) R(s) = 1/s
G(s)
9 / (s^2 + 9s + 9)
C(s)
Overdamped
c(t) = 1 + 0.171e^{-7.854t} − 1.171e^{-1.146t}
(c) R(s) = 1/s
G(s)
9 / (s^2 + 2s + 9)
C(s)
Underdamped
c(t) = 1 − e^{-t}(cos√8t + (√8/8) sin√8t) = 1 − 1.06e^{-t} cos(√8t − 19.47°)
(d) R(s) = 1/s
G(s)
9 / (s^2 + 9)
C(s)
Undamped
c(t) = 1 − cos 3t
(e) R(s) = 1/s
G(s)
9 / (s^2 + 6s + 9)
C(s)
Critically damped
c(t) = 1 − 3te^{-3t} − e^{-3t}
c(t)
t

- The input pole generates the constant forced response. The two system poles on the real axis generates the exponential response:

$$K_1 e^{-7.854t}, K_2 e^{-1.146t}$$

- Known as overdamped response.

Underdamped Response

- Given

$$C(s) = \frac{9}{s(s^2 + 2s + 9)}$$

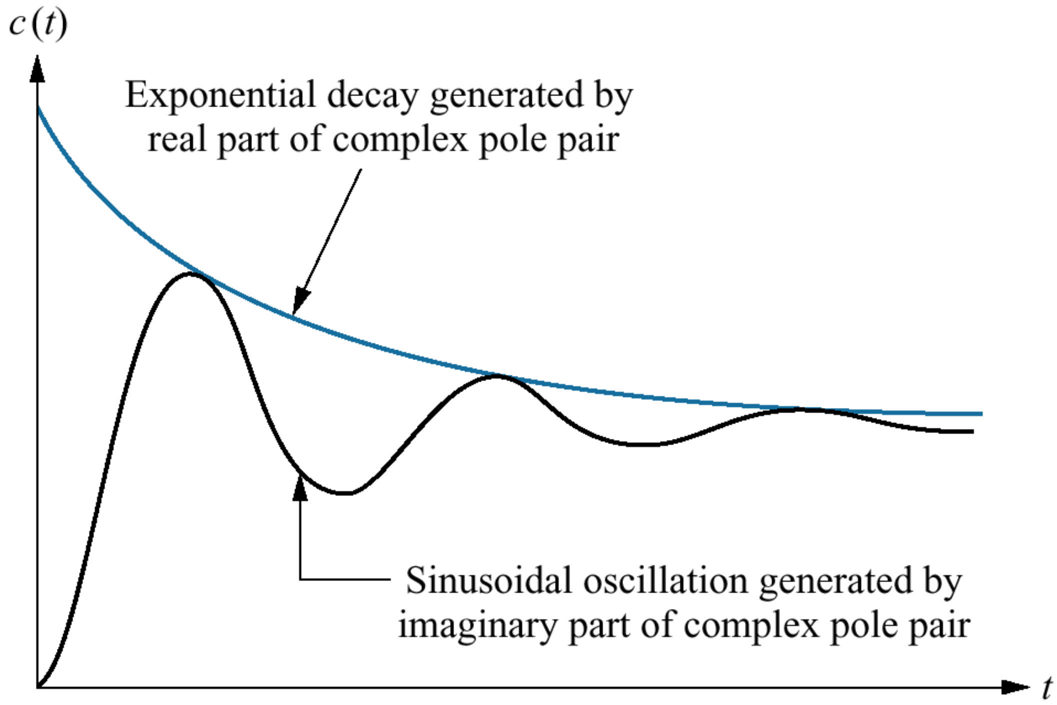
$$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t)$$

- Poles at:

$$s = 0 \text{ (due to input step)}$$

$$s = -1 \pm j\sqrt{8} \text{ (natural response)}$$

  - Real part of pole matches the exponential decay frequency of the sinusoid's amplitude.
  - Imaginary part matches frequency of the sinusoidal oscillation. Define this frequency as ω_d, i.e. damped frequency of oscillation, equals $\sqrt{8}$ in this case.



Undamped Response

- Given

$$C(s) = \frac{9}{s(s^2+9)}$$

$$c(t) = 1 - \cos 3t$$

- Two imaginary poles at $\pm j3$ generate a sinusoidal natural response with frequency $\omega = 3\,\text{rad/s}$.

- No real part implies an exponential that does not decay.

Critically Damped Response

- Given

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s+3)^2}$$

$$c(t) = 1 - 3te^{-3t} - e^{-3t}$$

- Pole at origin due to the step input and two real poles at p_1, p_2=-3 from the system itself.
- Response consists of exponential and exponential times *t*. Exponential parameter equals the location of the real poles.
- Responses are fastest possible without the overshoot.

Unstable

- Poles are located on the right half plane.
- Results in positive exponentials. Response goes to infinity as time increases.
- Example:

$$C(s) = \frac{9}{s(s-3)} = \frac{-3}{s} + \frac{3}{s-3}$$

$$c(t) = -3 + e^{3t}$$

Summary of Natural Responses

1. Overdamped responses:

   - Poles: two real at $-\sigma_1$, $-\sigma_2$.

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. Underdamped responses:

   - Poles: two complex at $-\sigma_d \pm j\omega_d$

$$c(t) = K_1 e^{-\sigma_d t}\left(K_2 \cos \omega_d t + K_3 \sin \omega_d t\right)$$

3. Undamped responses:

   - Poles: 2 imaginary at $\pm j\omega_1$

$$c(t) = A \cos \omega_1 t$$

4. Critically damped responses:

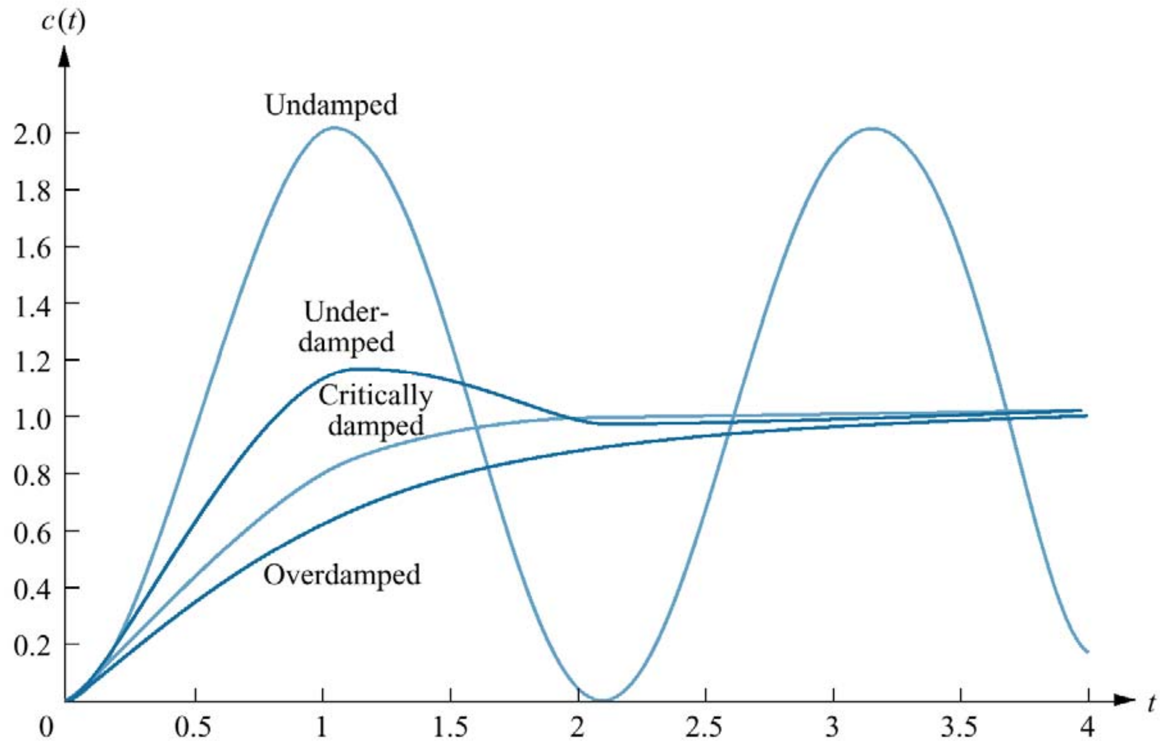
   - Poles: Two real at $-\sigma_1$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_2 t}$$

5. Unstable:

   - Poles: Any right half plane poles, e.g. at $\sigma_d \pm j\omega_d$.

$$c(t) = K_1 e^{\sigma_d t}\left(K_2 \cos \omega_d t + K_3 \sin \omega_d t\right)$$


c(t)
Undamped
2.0
1.8
1.6
1.4
Under-
damped
1.2
Critically
damped
1.0
0.8
0.6
Overdamped
0.4
0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
t