TIME RESPONSE

- This topic involves analysis of a control system in terms of time response and steady-state error.

- At the end of this topic, the students should be able to:
  1. Recognize standard input test signal.
  2. Find and sketch the response of first, second and third order systems.
  3. Identify the poles and zeros of a transfer function.
  4. State the important specifications of a time domain response.
  5. Identify and state the order, type and steady state error coefficient given a transfer function.
Standard Input Test Signal

- Test input signals are used, both analytically and during testing, to verify the design.

- Engineer usually select standard test inputs, as shown in Table 4.1: impulse, step, ramp, parabola and sinusoid.

- An impulse is infinite at $t=0$ and zero elsewhere. The area under the unit impulse is 1.

- A step input represents a constant command, such as position, velocity, or acceleration.

- Typically, the step input command is of the same form as the output. For example, if the system’s output is position, the step input represents a desired position, and the output represents the actual position.

- The designer uses step inputs because both the transient response and the steady-state response are clearly visible and can be evaluated.

- The ramp input represents a linearly increasing command. For example, if the system’s output is position, the input ramp represents a linearly increasing position.
### Table 4.1

<table>
<thead>
<tr>
<th>Input</th>
<th>Function</th>
<th>Description</th>
<th>Sketch</th>
<th>Use</th>
</tr>
</thead>
</table>
| Impulse | $\delta(t)$ | $\delta(t) = \infty$ for $0^- < t < 0^+$  
= 0 elsewhere  
$\int_{0^-}^{0^+} \delta(t) \, dt = 1$ | $f(t)$ | Transient response         |
|         |          |                                                                             |        | Modeling                   |
| Step    | $u(t)$   | $u(t) = 1$ for $t > 0$  
= 0 for $t < 0$ | $f(t)$ | Transient response         |
|         |          |                                                                             |        | Steady-state error         |
| Ramp    | $tu(t)$  | $tu(t) = t$ for $t \geq 0$  
= 0 elsewhere | $f(t)$ | Steady-state error         |
| Parabola | $\frac{1}{2} t^2 u(t)$ | $\frac{1}{2} t^2 u(t) = \frac{1}{2} t^2$ for $t \geq 0$  
= 0 elsewhere | $f(t)$ | Steady-state error         |
| Sinusoid | $\sin \omega t$ |                                                                             | $f(t)$ | Transient response         |
|         |          |                                                                             |        | Modeling                   |
|         |          |                                                                             |        | Steady-state error         |
• The response to an input ramp test signal yields additional information about the steady-state error.

• The previous discussion can be extended to parabolic inputs, which are also used to evaluate a system’s steady-state error.

• Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model (Frequency Response).

**Poles, Zeros and System Response**

• The output response of a system is the sum of two responses: the *forced response* (steady-state response) and the *natural response*.

• It is possible to relate the poles and zeros of a system to its output response.

• The *poles* of a transfer function are:
  
  o The values of the Laplace transform variable, $s$, that cause the transfer function to become infinite.

  o Any roots of the denominator of the transfer function that are common to roots of the numerator.
The zeros of a transfer function are:

- The values of the Laplace transform variable, \( s \), that cause the transfer function to become zero.
- Any roots of the numerator of the transfer function that are common to roots of the denominator.

**Poles and Zeros of a First-Order System**

Example: Given \( \frac{C(s)}{R(s)} = \frac{(s + 2)}{(s + 5)} \). Find the pole and zero of the system and plot on the \( s \) plane. Find the unit step response of the system.
\[ R(s) = \frac{1}{s} \]

\[ G(s) = \frac{s + 2}{s + 5} \]

\[ C(s) \]

\[ j\omega \]

\[ s \text{-plane} \]

\[ \sigma \]

(a)

(b)

Input pole

\[ \frac{1}{s} \]

S-plane

\[ j\omega \]

\[ \sigma \]

System zero

\[ s + 2 \]

S-plane

\[ j\omega \]

\[ \sigma \]

System pole

\[ \frac{1}{s + 5} \]

S-plane

\[ j\omega \]

\[ \sigma \]

\[ C(s) = \frac{2/5}{s} + \frac{3/5}{s + 5} \]

\[ c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t} \]

Output transform

Output time response

(c)

Forced response  Natural response
• Example conclusions:
  o A pole of the input function generates the form of the *forced response*.
  o A pole of the transfer function generates the form of the *natural response* (the pole at \(-5\) generated \(e^{-5t}\)).
  o A pole on the real axis generates an *exponential* response of the form \(e^{-\alpha t}\), where \(-\alpha\) is the pole location on the real axis.
  o The farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
  o The zeros and poles generate the *amplitudes* for both the forced and natural responses.

**First-Order Systems**

• Consider a first order system, \(\frac{C(s)}{R(s)} = \frac{a}{s + a}\) with a unit step input

![Diagram of first-order system](attachment:diagram.png)
- Time response:

\[ c(t) \]

Initial slope = \( \frac{1}{\text{time constant}} = a \)

63% of final value at \( t = \text{one time constant} \)

\[
\begin{align*}
\text{Initial slope:} & \quad \frac{1}{a} \\
\text{63% of final value:} & \quad \frac{2}{a} \\
\end{align*}
\]

\[
\begin{align*}
T_r & \quad \text{time constant} \\
T_s & \quad \text{time constant} \\
\end{align*}
\]
• At \( t = 1/a \),

**Specification: Time Constant**

• The Time constant, \( T_C \) is the time for the step response to rise to 63% of its final value.

• From the response, \( T_C = 1/a \).

• The bigger \( a \) is, the farther it will be from the imaginary axis and the faster the transient response will be.

**Specification: Rise Time**

• Rise time, \( T_r \) is the time it takes for the response to go from 0.1 to 0.9 of its final value.
Specifications: Settling Time

- Settling time, $T_S$ is the time for the response to reach and stay within, 2% of its final value.

First-Order Transfer Functions via Testing

- Often it is not possible or practical to obtain a system’s transfer function analytically (i.e. using techniques from last topic).

- We can have an approximation of a system’s transfer function by looking at its step response.

- The actual system is fed with a step input, and its response recorded experimentally.

- Given that the step response is known, the transfer function can be found.

- Consider a simple first order system:
\[ G(s) = \frac{K}{(s + a)} \]

For step response, \( C(s) = \frac{K}{s(s + a)} = \frac{K}{s} - \frac{K}{s + a} \)

\[ \Rightarrow c(t) = \frac{K}{a} - \frac{K}{a} e^{-at} \]

- Hence, if we can identify \( K \) and \( a \) from laboratory testing, we can obtain the transfer function of the system.

- **Example:** Find the transfer function for the step response below:
- **Example**: A system has a transfer function,

$$G(s) = \frac{50}{s + 50}.$$  

Find the time constant, settling time and rise time. Sketch the unit step response of the system.
• **Example:** Sketch the unit step response for a system with transfer function, \( G(s) = \frac{100}{s + 50} \).