YOGYAKARTA STATE UNIVERSITY

MATHEMATICS AND NATURAL SCIENCES FACULTY

MATHEMATICS EDUCATION STUDY PROGRAM

Topic: Linear Systems

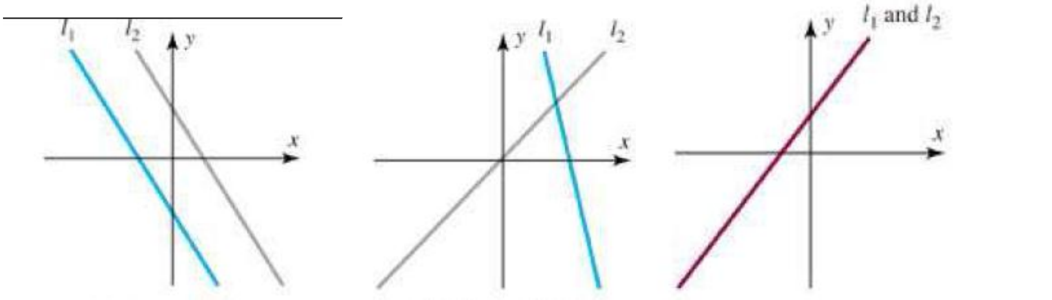
An equation of the type $a_1x_1 + a_2x_2 + \ldots + a_nx_n = \bar{b}$, (1)
expressing $\bar{b}$ in terms of the variables $x_1, x_2, \ldots, x_n$ and the known constants $a_1, a_2, \ldots, a_n$, is called a linear equation. A solution to the linear equation (1) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are subtituted in (1).

Example 1: $x_1 = 2, x_2 = 3, \text{and } x_3 = -4$ is a solution to the linear equation $6x_1 - 3x_2 + 4x_3 = -13$, because $6(2) - 3(3) + 4(-4) = -13$. This is not the only solution to the given linear equation, since $x_1 = 3, x_2 = 1, \text{dan } x_3 = -7$ is another solution.

A system of equations that has no solutions is said to be ***inconsistent***; if there is at least one solution of the system, it is called ***consistent***. To illustrate the possibilities that can occur in solving systems of linear equations, consider a general system of two linear equations in the unknowns *x* and *y*:

$$\begin{cases} a_1x + b_1y = c_1 & (a_1, b_1 \text{ not both zero}) \\ a_2x + b_2y = c_2 & (a_2, b_2 \text{ not both zero}) \end{cases}$$

The graphs of these equations are lines; call them l_1 and l_2. Since a point (*x*, *y*) lies on a line if and only if the numbers *x* and *y* satisfy the equation of the line, the solutions of the system of equations correspond to points of intersection of l_1 and l_2. There are three possibilities, illustrated in Figure 1:



(a) No solution (b) one solution (c) infinitely many solutions

The lines l_1 and l_2 may be parallel, in which case there is no intersection and consequently no solution to the system. The lines l_1 and l_2 may intersect at only one point, in which case the system has exactly one solution. The lines l_1 and l_2 may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system.

Although we have considered only two equations with two unknowns here, we will show later that the same three possibilities hold for arbitrary linear systems

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

More generally, a linear system is a system of m linear equations in n unknowns, and can be conveniently denoted by

$$\begin{cases} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = & b_1 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + & a_{mn}x_n = & b_m . \end{cases} \quad (2)$$

A solution to a linear system (2) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that each equation in (2) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are subtituted in (2).

The Solution of The Linear System

The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but is easier to solve. This new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically:

The Linear Equation Operation

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another.

Example 2: Given the linear system:

$$\begin{cases} x + \quad y + 2z = 9 \\ 2x + \ 4y - 3z = 1 \\ 3x + \ 6y - 5z = 0 \end{cases}$$

To eliminate x from the second equation, we add (-2) times the first equation to the second one to obtain

$$\begin{cases} x + \quad y + 2z = 9 \\ \qquad 4y - 3z = -17 \\ 3x + \ 6y - 5z = 0 \end{cases}$$

To eliminate x from the second equation, we add (-3) times the first equation to the third one to obtain

$$\begin{cases} x + \quad y + \ 2z = 9 \\ \quad 2y - \ 7z = -17 \\ \quad 3y - 11z = -27 \end{cases}$$

Multiply the second equation with $1/2$, to obtain

$$\begin{cases} x + \quad y + \ 2z = 9 \\ \qquad y - \frac{7}{2}z = -\frac{17}{2} \\ \quad 3y - 11z = -27 \end{cases}$$

Added (-3) times the second equation to the third one to obtain

$$\begin{cases} x + \ y + \ 2z = 9 \\ \quad y - \frac{7}{2}z = -\frac{17}{2} \\ \qquad - \frac{1}{2}z = -\frac{3}{2} \end{cases}$$

Multiply the third equation with -2 , to obtain

$$\begin{cases} x + \ y + \ 2z = 9 \\ \quad y - \frac{7}{2}z = -\frac{17}{2} \\ \qquad\qquad z = 3 \end{cases}$$

Added (-1) times the second equation to the first one to obtain

$$\begin{cases} x \qquad + \frac{11}{2}z = \frac{35}{2} \\ \quad y - \frac{7}{2}z = -\frac{17}{2} \\ \qquad\qquad z = 3 \end{cases}$$

Added $(-\frac{11}{2})$ times the third equation to the first one and $(\frac{7}{2})$ times the third equation to the second one to obtain

$$\begin{cases} x \quad\quad\quad = 1 \\ \quad\; y \quad\; = 2 \\ \quad\quad\; z = 3 \end{cases}$$

We conclude that the solution of the linear system is $x = 1, y = 2, z = 3$.

Exercises 1:

1. Which of the following are linear equations in x_1, x_2 and x_3?

   (a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$

   (b) $x_1 + 3x_2 + x_1x_3 = 2$

   (c) $x_1 = -7x_2 + 3x_3$

   (d) $x_1^{-2} + x_2 + 8x_3 = 5$

   (e) $x_1^{3/5} - 2x_2 + x_3 = 4$

   (f) $\pi x_1 - \sqrt{2}x_2 + \dfrac{1}{3}x_3 = 7^{1/3}$

2. Find the solution set of each of the following linear equations.

   (a) $7x - 5y = 3$.

   (b) $3x_1 - 5x_2 + 4x_3 = 7$

   (c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

   (d) $3v - 8w + 2x - y + 4z = 0$.

3. Find the solution of given the linear system.

   (a) $\begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$ (b) $\begin{cases} 2x + 2z = 1 \\ 3x - y + 4z = 7 \\ 6x + 2y - z = 0 \end{cases}$

4. (a) Find a linear equation in the variables x and y that has the general solution $x = 5 + 2t,\ y = t$.

(b) Show that $x = t$, $y = \frac{1}{2}t - \frac{5}{2}$ is also the general solution of the equation in part (a).

5. Consider the system of equations

$$\begin{cases} x + y + 2z = a \\ x + z = b \\ 2x + y + 3z = c \end{cases}$$

Show that for this system to be consistent, the constant *a*, *b*, and *c* must satisfy *c* = *a* + *b*.

6. Show that if the linear equations $x_1 + kx_2 = c$ and $x_1 + lx_2 = d$ have the same solution set, then the equations are identical.

7. For which value(s) of the constant *k* does the system

$$\begin{cases} x - y = 3 \\ 2x - 2y - k \end{cases}$$

have no solutions? Exactly one solution? Infinitely many solutions? Explain your reasoning.

8. Consider the system of equations

$$\begin{cases} ax + by = k \\ cx + dy = l \\ ex + fy = m \end{cases}$$

Indicate what we can say about the relative positions of the lines $ax + by = k$, $cx + dy = l$, and $ex + fy = m$ when

(a) the system has no solutions.

(b) the system has exactly one solution.

(c) the system has infinitely many solutions.

9. If the system of equations in Exercise 8 is consistent, explain why at least one equation can be discarded from the system without altering the solution set.

10. If *k* = *l* = *m* = 0 in Exercise 8, explain why the system must be consistent. What can be said about the point of intersection of the three lines if the system has exactly one solution?

Topic: Matrices

Definition 1. An $m \times n$ matrix A is a rectangular of mn real (or complex) numbers arranged in m horizontal rows and n vertical columns :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \qquad (3)$$

The ith row of A, $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$, $(1 \le i \le m)$; the jth column of A, $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$,

$(1 \le j \le n)$.

If $m = n$, we say that A is a square matrix of order n, and that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal A. We refer to the number a_{ij}, which is in the ith row and jth column of A, and (3) often write as $A = [a_{ij}]$. Sum of all main diagonal is called ***trace*** A (tr A).

Example 3: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ is an 2×3 matrix with

$a_{12} = 2, a_{13} = 3, a_{22} = 0, a_{23} = 1$. $B = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$ is an 2×2 matrix, with

$tr(A) = 1 + (-3) = (-2)$. $C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is an 3×1 matrix or column matrix.

$D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ is an 3×3 matrix, with $tr(A) = 3$. $E = [3]$ is an 1×1 matrix.

$F = [-1 \ \ 0 \ \ 2]$ is an 1×3 matrix or row matrix.

The Equality of Two Matrices

Definition 2. Two $m\times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if $a_{ij} = b_{ij}$, $1 \le i \le m, 1 \le j \le n$, that is, if corresponding elements are equal.

Example 4: The matrices

$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$ are equal if $w = -1, x = -3, y = 0$ and $z = 5$.

Matrix Addition

Definition 3. If A = [a_{ij}] and B = [b_{ij}] are $m\times n$ matrices, then A+B is a $m\times n$ matrix C =[c_{ij}], defined by c_{ij} = a_{ij} + b_{ij} $(1 \le i \le m, 1 \le j \le n)$ and the difference between A and B is D = [d_{ij}] noted as $A + (-1)B = A - B$, defined by $d_{ij} = a_{ij} - b_{ij}$.

Example 5:

1. Given $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix}$.

   Then $A + B = \begin{bmatrix} 1+0 & -2+2 & 4+(-4) \\ 2+1 & -1+3 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \end{bmatrix}$.

2. If $A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}$, then

$$A-B=\begin{bmatrix} 2-2 & 3+1 & -5-3 \\ 4-3 & 2-5 & 1+2 \end{bmatrix}=\begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}.$$

3. Given $A=\begin{bmatrix} 4 & 8 \\ 2 & 14 \end{bmatrix}$, $B=\begin{bmatrix} 6 & -2 \\ 10 & 4 \end{bmatrix}$ and $C=\begin{bmatrix} -10 & 4 \\ 21 & -3 \end{bmatrix}$. Then

$$A + B - C=\begin{bmatrix} 4+6+10 & 8-2-4 \\ 2+10-21 & 14+4+3 \end{bmatrix}=\begin{bmatrix} 20 & 2 \\ -9 & 21 \end{bmatrix}.$$

Additon Matrix Properties

Given *A*, *B*, *C*, and *D* are $m \times n$ matrices.

(a) $A + B = B + A$.

(b) $A + (B + C) = (A + B) + C$.

(c) $A + O = A$. O matrix is called ***zero matrix*** or ***the addition identity***.

(d) $A + D = O$. $D = -A$ is called ***the inverse addition matrix*** or ***the negative of*** *A*.

Example 6:

1. Given $A=\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$ and $O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$A+O=\begin{bmatrix} 4+0 & -1+0 \\ 2+0 & 3+0 \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}=A.$$

2. Given $A=\begin{bmatrix} 2 & 3 & 4 \\ -4 & 5 & -2 \end{bmatrix}$. Then $-A=\begin{bmatrix} -2 & -3 & -4 \\ 4 & -5 & 2 \end{bmatrix}$, and we obtained

$$A+(-A)=O.$$

Scalar Multiplication

Definition 4. If $A=[a_{ij}]$ is $m \times n$ matrix and *r* is a real number, then scalar multiple of *A* by *r*, *rA*, is the $m \times n$ matrix $B=[b_{ij}]$, where $b_{ij} = ra_{ij}$ $(1 \le i \le m, 1 \le j \le n)$.

Example 7:

1. If $r = -3$ and $A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & -5 & 0 \\ 3 & 6 & -2 \end{bmatrix}$, then

$$rA = -3\begin{bmatrix} 4 & -2 & 3 \\ 2 & -5 & 0 \\ 3 & 6 & -2 \end{bmatrix} = \begin{bmatrix} (-3)(4) & (-3)(-2) & (-3)(3) \\ (-3)(2) & (-3)(-5) & (-3)(0) \\ (-3)(3) & (-3)(6) & (-3)(-2) \end{bmatrix} = \begin{bmatrix} -12 & 6 & -9 \\ -6 & 15 & 0 \\ -9 & -18 & 6 \end{bmatrix}.$$

2. Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$. Then

$$A - 3B + 5C = \begin{bmatrix} 1-12+5 & 2-9-15 \\ 3-6+10 & 4-3-20 \end{bmatrix} = \begin{bmatrix} -6 & -22 \\ 7 & -19 \end{bmatrix}.$$

3. Given $A = \begin{bmatrix} -a & b \\ -b & 2a \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ b & 2a \end{bmatrix}$, and any scalar k . Then

$$3kA - 2kB = \begin{bmatrix} -3ka & 3kb \\ -3kb & 6ka \end{bmatrix} - \begin{bmatrix} 2ka & 2kb \\ 2kb & 4ka \end{bmatrix} = \begin{bmatrix} -5ka & kb \\ -5kb & 2ka \end{bmatrix} = k\begin{bmatrix} -5a & b \\ -5b & 2a \end{bmatrix}.$$

Scalar Multiplication Matrix Properties

If r and s are real number and A and B are matrices, then

(a) $r(sA) = (rs)A$. (c) $r(A + B) = rA + rB$.

(b) $(r + s)A = rA + sA$. (d) $A(rB) = r(AB) = (rA)B$.

Example 8: Given $r = -2$, $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 0 & -2 \end{bmatrix}$. Then

$$A(rB) = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ -2 & -8 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 8 & 0 \end{bmatrix}, \text{ and}$$

$$r(AB) = (-2)\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 8 & 0 \end{bmatrix}.$$

Matrix Multiplication

Definition 5. If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is an $p \times n$ matrix, then the product of A and B, denoted AB, is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj} \quad (1 \le i \le m, 1 \le j \le n).$$

Example 9:

1. Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} (1)(-2)+(2)(4)+(-1)(2) & (1)(5)+(2)(-3)+(-1)(1) \\ (3)(-2)+(1)(4)+(4)(2) & (3)(5)+(1)(-3)+(4)(1) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}.$$

2. Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$. If $AB=C$ then the (3,2) entry of AB is c_{32}, which is $R_3(A)$ x $C_2(B)$. We now have

$$R_3(A) \cdot C_2(B) = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = -5.$$

3. Given $A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}$. If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, then x and y are,

$$AB = \begin{bmatrix} 2+4x+3y \\ 4-4+y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$ Then $2+4x+3y = 12$ then $y = 6$, we have $x = -2$ and $y = 6$.

4. Given $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}$ but $BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}$.

   We conclude $AB \ne BA$.

5. Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. Then the second column of AB is

$$A \cdot C_2(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix}.$$

6. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$.

7. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \\ i & j \end{bmatrix}$, then *AB* undefined, because *A* and *B* ukurannya tidak sesuai.

Matrix Multiplication Properties

If *A*, *B*, and *C* have the suitable size, then

(a) $A(BC) = (AB)C$. (b) $A(B + C) = AB + AC$. (c) $(A + B)C = AC + BC$.

Example 10:

1. Given $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}$.

   Then $A(BC) = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$

   and $(AB)C = \begin{bmatrix} 19 & -1 & 6 & 13 \\ 16 & -8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$.

2. Given $A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 2 & -2 \end{bmatrix}$.

Then $A(B+C)=\begin{bmatrix}2 & 2 & 3\\ 3 & -1 & 2\end{bmatrix}\begin{bmatrix}0 & 2\\ 3 & 2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}21 & -1\\ 7 & -2\end{bmatrix}$

and $AB+AC=\begin{bmatrix}15 & 1\\ 7 & -4\end{bmatrix}+\begin{bmatrix}6 & -2\\ 0 & 2\end{bmatrix}=\begin{bmatrix}21 & -1\\ 7 & -2\end{bmatrix}$.

3. Given $A=\begin{bmatrix}4 & -2 & 3\\ 5 & 0 & 2\end{bmatrix}$. Then $I_2 A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}4 & -2 & 3\\ 5 & 0 & 2\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 5 & 0 & 2\end{bmatrix}=A.$

   And $AI_3=\begin{bmatrix}4 & -2 & 3\\ 5 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 5 & 0 & 2\end{bmatrix}$.

Notes :

1. If $AB = AC$, then uncertain $B = C$.
2. If $AB = O$, then uncertain $A = O$ or $B = O$.

Example 11:

1. If $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}4 & -6\\ -2 & 3\end{bmatrix}$, then $AB=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. So A and B are not nol matrix but $AB = O$.
2. If $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$, and $C=\begin{bmatrix}-2 & 7\\ 5 & -1\end{bmatrix}$, then

   $AB=AC=\begin{bmatrix}8 & 5\\ 16 & 10\end{bmatrix}$, but $B \neq C$.

The Transpose of Matrix

Definition 6. If $A = [a_{ij}]$ is an $m \times n$ matrix, then $A^T = \left[a_{ij}^T\right]$, where $a_{ij}^T = a_{ji}$, $1 \le i \le m$, $1 \le j \le n$, is called ***the transpose of*** A.

Example 12:

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \\ 0 & 4 & 3 \end{bmatrix}, \text{ then } B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 2 & -3 \end{bmatrix}, \text{ then } C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & -5 & 1 \end{bmatrix}, \text{ then } D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \text{ then } E^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$$

The Transpose Properties

If r is scalar number and A and B are matrices, then

(a) $(A^T)^T = A$.

(b) $(A + B)^T = A^T + B^T$.

(c) $(AB)^T = B^TA^T$.

(d) $(rA)^T = rA^T$.

Example 13: Given $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$.

Then $(AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$ and $B^T A^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$.

Definition 7. A matrix $A = [a_{ij}]$ is called ***symmetric*** if $A^T = A$.

Example 14: The matrices are symmetric,

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix} \text{ and } I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Exercises 2:

1. Suppose that *A*, *B*, *C*, *D*, and *E* are matrices with the following sizes: $A(4\times 5)$, $B(4\times 5)$, $C(5\times 2)$, $D(4\times 2)$, $E(5\times 4)$

   Determine which of the following matrix expressions are defined. For those that are defined, give the size of the resulting matrix.

   (a)BA

   (b)$AC + D$

   (c)$AE + B$

   (d)$AB + B$

   (e)$E(A + B)$

   (f) $E(AC)$

   (g)E^TA

   (h)$(A^T + E)D$

2. Given r = -4 and $A=\begin{bmatrix}1 & 3 & 2\\ 2 & 1 & -3\end{bmatrix}$, $B=\begin{bmatrix}4 & 2 & -1\\ -2 & 1 & 5\end{bmatrix}$.

   Show that : (a) $(A + B)^T = A^T + B^T$ (b) $(rA)^T = rA^T$.

3. Given $A=\begin{bmatrix}1 & 3 & 2\\ 2 & 1 & -3\end{bmatrix}$, $B=\begin{bmatrix}3 & -1\\ 2 & 4\\ 1 & 2\end{bmatrix}$. Show that $(AB)^T = B^TA^T$.

4. Given

$$A=\begin{bmatrix}2 & 1 & -2\\ 3 & 2 & 5\end{bmatrix},\ B=\begin{bmatrix}2 & -1\\ 3 & 4\\ 1 & -2\end{bmatrix},\ C=\begin{bmatrix}2 & 1 & 3\\ -1 & 2 & 4\\ 3 & 1 & 0\end{bmatrix},\ D=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$$

$$E=\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 3\\ -3 & 2 & -1\end{bmatrix}, \text{ and } F=\begin{bmatrix}1 & 0\\ 2 & -3\end{bmatrix}.$$

If possible compute :

(a) $(AB)^T$ (b) B^TA^T (c) A^TB^T (d) BB^T

(e) B^TB

(f) $(3C - 2E)^TB$ (g) $A^T(D + F)$ (h) $B^TC + A$

(i) $(2E)A^T$ (j) $(B^T + A)C$.

5. Determine a constant k such that $(kA)^T(kA) = 1$, where $A = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$.

For no. 6 – 7, given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$$

If possible compute :

(a). A^T and $(A^T)^T$ (c). $(2D + 3F)^T$ (e). $2A^T + B$

(b). $(C + E)^T$ and $C^T + E^T$ (d). $D - D^T$ (f). $(3D - 2F)^T$

If possible compute :

(a). $(2A)^T$ (c). $(3B^T - 2A)^T$ (e). $(-A)^T$ and $(A)^T$

(b). $(A - B)^T$ (d). $(3A^T - 5B^T)^T$ (f). $(C + E + F^T)^T$

Types of Matrices

A. Identity Matrix

Example 15:

1. **Upper Triangular Matrix**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

2. **Lower Triangular Matrix**

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ -3 & 1 & 3 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & -2 & 3 \end{bmatrix}$$

3. **Diagonal Matrix** $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

4. **Scalar Matrix** $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

5. **Identity Matrix** $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. Square Matrices

Example 16:

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ then $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A and

   B are called ***commutative matrices***.

2. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $C = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ then $AB = -BA$, $AC = -CA$,

   and

   $BC = -CB$. A, B, and C are called ***not commutative matrices*** .

3. A is a ***periodic matrix*** with period 2, because

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \text{ maka } A^2 = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix} \text{ dan } A^{2+1} = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}.$$

4. A and B are ***idempotent matrices***, cause

$$\text{if then } B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \text{ then } B^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = B .$$

5. A is called ***nilpotent matrix*** with index 2, because

$$\text{if } A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \text{ then } A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O .$$

6. A and B are called ***involuntary matrices***, because

$$\text{if } A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \text{ then } A^2 = I_3 \text{, and } B = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix} \text{ then } B^2 = I_3 .$$

C. **Hermitian Matrix**

Example 17: Given

$$A = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix} \text{ then}$$

1. A is a ***Hermitian matrix***, because $A^t = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 0 \end{bmatrix}$ and $\overline{A}^t = A$

2. B is a ***Skew Hermitian Matrix***,

   because $B^t = \begin{bmatrix} i & -1+i & -2-3i \\ 1+i & 2i & -1 \\ 2-3i & 1 & 0 \end{bmatrix}$ and $\overline{B}^t = -B$

3. iB is a ***Hermitian Matrix***, because

   $$(iB)^t = \begin{bmatrix} -1 & -i-1 & -2i+3 \\ i-1 & -2 & -i \\ 2i+3 & i & 0 \end{bmatrix} \text{ and } (i\overline{B})^t = iB$$

4. $\overline{A}$ is a ***Hermitian Matrix***, because

$$\overline{A} = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & +i \\ 2+3i & -i & 0 \end{bmatrix}, \text{ and } \overline{A}^t = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix} \text{ then } \overline{\overline{(A)}}^t = \overline{A}.$$

5. $\overline{B}$ is a ***skew Hermitian matrix***, because

$$\overline{B} = \begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & 1 \\ -2+3i & -1 & 0 \end{bmatrix}, \text{ and } \overline{B}^t = \begin{bmatrix} -i & -1-i & -2+3i \\ 1-i & -2i & -1 \\ 2+3i & 1 & 0 \end{bmatrix} \text{ then } \overline{\overline{B}}^t = -\overline{B}$$

YOGYAKARTA STATE UNIVERSITY

MATHEMATICS AND NATURAL SCIENCES FACULTY

MATHEMATICS EDUCATION STUDY PROGRAM

Topic: Augmented Matrix

An arbitrary system of m linear equations in n unknowns can be written as,

$$\begin{cases} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = & b_1 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + & a_{mn}x_n = & b_m. \end{cases} \tag{4}$$

where x_1, x_2, $\ldots, x_n$ are the unknowns and the a_{ij} and b_i, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ denote constants.

This system can be abbreviated by writing in matrix term

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \overline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{or } A\overline{x} = \overline{b}.$$

With A is a coefisien matrix and $\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called an augmented matrix (4).

Example 18: The augmented matrix for the system of 3 linear equations and 3 unknowns

$$\begin{cases} -2x \quad\quad + \quad z = 5 \\ 2x \ + 3y - \ 4z = 7 \\ 3x \ + 2y + \ 2z = 3. \end{cases}$$

With $A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -4 \\ 3 & 2 & 2 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\bar{b} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$, is $\left[\begin{array}{ccc:c} -2 & 0 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right]$.

Example 19: The Matrix $\left[\begin{array}{ccc:c} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$ is an augmented matrix of linear system

$$\begin{cases} 2x \ - y + 3z = 4 \\ 3x \quad\ \ + 2z = 5. \end{cases}$$

Exercises 3:

1. Let see a linear system

$$\begin{cases} 2x + \quad\quad\quad\quad\quad w = \ \ 7 \\ 3x + \ 2y + 3z \quad\ = -2 \\ 2x + \ 3y - 4z \quad\ = \ \ 3 \\ x + \quad\quad 3z \quad\ = \ \ 5. \end{cases}$$

   (a) Find the coefisien matrix. (b) Find the augmented matrix.

   (c) Write the system on matrix term.

2. Write the linear system of this augmented matrix : $\left[\begin{array}{cccc:c} -2 & -1 & 0 & 4 & 5 \\ -3 & 2 & 7 & 8 & 3 \\ 1 & 0 & 0 & 2 & 4 \\ 3 & 0 & 1 & 3 & 6 \end{array}\right]$.

3. Write the linear system of this augmented matrix : $\left[\begin{array}{ccc:c} 2 & 0 & -4 & 3 \\ 0 & 1 & 2 & 5 \\ 1 & 3 & 4 & -1 \end{array}\right]$.

4. Given the linear system :

$$\begin{cases} 3x & -y & + & 2z & = 4 \\ 2x & +y & & & = 2 \\ & y & + & 3z & = 7 \\ 4x & & - & z & = 4. \end{cases}$$

(a) Find the coefisien matrix. (b) Find the augmented matrix.

(c) Write the linear system on matrix term.

5. Find the answer with elimination methods

$$\begin{cases} x + 3y = 9 \\ 2x + y = 8 \end{cases}.$$

6. Find the answer (if any) of this linear system

$$\begin{cases} x - 2y + z = 2 \\ 2x - y - 4z = 13 \\ x - y - z = 5 \end{cases}.$$

7. Find *a* and *b*, so this linear system has

   (i). exactly one solution (ii). no solution (iii). Infinitely many solution

   (a). $\begin{cases} x - 2y = 1 \\ ax + by = 5 \end{cases}$ (b). $\begin{cases} x + by = -1 \\ ax + 2y = 5 \end{cases}$ (c). $\begin{cases} x - by = -1 \\ x + ay = 3 \end{cases}.$

8. Given the linear system

$$\begin{cases} ax + by = p \\ cx + dy = q \\ ex + fy = r \end{cases}$$

   When the lines of the 3 linear eqution makes

   (i). the system has exactly one solution,

   (ii). the system has no solution,

   (iii). the system has infinitely many solution.

9. For number 8, if the equation systems consistent, then one of the equations can be eliminated without changing the solution set.

10. Consider the equation system:

$$\begin{cases} x + y + 2z = a \\ x \quad\;\; + z = b \\ 2x + y + 3z = c \end{cases}$$

Suppose the system is consistent then *a*, *b*, and *c* has satisfy $c = a + b$.

11. Prove: If the linear system $x_1 + kx_2 = c$ and $x_1 + lx_2 = d$ has the same solution set, then the equations are identic.