

# The Idempotent Regular Elements of Semigroups

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## Abstract

In this paper, we will generalize the characteristics of the idempotent elements of a regular semigroup and the idempotent elements of its quotient semigroup. Based on the Characteristics of the idempotent elements of an arbitrary semigroup, we obtain that : For an arbitrary semigroup  $S$  and a congruence  $\rho$  on  $S$ , then for every  $(a)\rho \in E(S/\rho)$  there is  $e \in E(S)$  such that  $(a)\rho = (e)\rho$ . Let  $\varphi : S \rightarrow T$ , be homomorphism, so we obtain : for every  $f \in E(R(\varphi))$  there is an element  $e \in E(S)$  such that  $(e)\varphi = f$ . The result of the generalization : for every idempotent regular element  $(a)\rho \in S/\rho$ , with  $S$  is an arbitrary semigroup ( must not be a regular semigroup ), there is  $e \in E(S)$ , such that  $(a)\rho = (e)\rho$ . As a corollary , we have : Let  $S, T$  be arbitrary semigroups, if  $\varphi : S \rightarrow T$  is a homomorphism, then for every idempotent regular element  $f \in E(R(\varphi))$  there is an element  $e \in E(S)$  such that  $(e)\varphi = f$ .

**Key words** : regular element, idempotent element, idempotent regular element.

## 1. Introduction

In the previous paper [3] has been proved the Characteristics of an idempotent element in  $M(e_0Se_0; I, \Lambda; P)/\rho$ , i.e.:

**Theorem 1.** *Let  $\rho$  be a congruence on  $M(e_0Se_0; I, \Lambda; P)$ , i.e. a Rees Matrix semigroup over a Bilinear Form semigroup  $S$ . If  $(u, e, f)\rho$  is an idempotent element in  $M(e_0Se_0; I, \Lambda; P)/\rho$ , then there is an idempotent element  $(t, e, f) \in M(e_0Se_0; I, \Lambda; P)$ , such that :  $(u, e, f)\rho = (t, e, f)\rho$ .*

This result can be generalized as follow [4]:

**Theorem 2.** *For every  $a \in E(S)$ , with  $S$  is a Bilinear Form Semigroup, there is  $(t, e, f)\rho \in E(M(e_0Se_0; I, \Lambda; P)/\rho)$ , such that  $(t, e, f)\varphi = a$ , with  $\rho$  is a congruency on  $M(e_0Se_0; I, \Lambda; P)$ .*

We can prove that  $S$  and  $M(e_0Se_0; I, \Lambda; P)$  are regular semigroups [2]. Hence, every element of these semigroups is a regular element. Based on the theory of the semigroup homomorphism, there is an injective homomorphism from the quotient semigroup to the image of the homomorphism.

From these conditions, in this paper, we will study about the generalization of the previous theorems, i.e. for the idempotent element of a regular semigroup and the homomorphism from a regular semigroup, the idempotent regular of a semigroup (*must not be a regular semigroup*) and the semigroup homomorphism.

## 2. Theoretical review :

These definitions are referred from [1]. Let  $S$  be a semigroup, an element  $e \in S$  is called an idempotent if and only if (next, be denoted by *Iff*)  $e^2 = e$ . An element  $a$ , in a semigroup  $S$ , is called regular iff there is  $x \in S$  such that  $axa = a$ . A semigroup  $S$  is called regular semigroup iff every element of  $S$  is regular. An element  $a' \in S$  is called an inverse of  $a$  iff  $ada = a$  and  $a'aa' = a'$ .

Let  $(S, \cdot)$ ,  $(T, \cdot)$  be semigroups. A map  $\phi: S \rightarrow T$  is called a semigroup homomorphism iff  $(\forall x, y \in S)(xy)\phi = ((x)\phi)((y)\phi)$ . Next,  $R(\phi) = \{(s)\phi: s \in S\} \subset T$  is called the range of  $\phi$ . The kernel of  $\phi$ , denoted by  $\ker \phi$ , is defined by  $\ker \phi = \{(a, b) \in S \times S \mid (a)\phi = (b)\phi\}$ .

Let a relation  $\rho$  be an equivalency relation on a set  $X$ , the notation  $(x, y) \in \rho$  is written as  $x\rho y$  or  $x \equiv y(\text{mod } \rho)$ . A set  $(x)\rho = \{y \in X \mid (x, y) \in \rho\}$  is a equivalency class. A relation  $\rho$  on a semigroup  $S$  is called compatible iff  $(\forall a, b, c, d \in S)(a, b) \in \rho$  and  $(c, d) \in \rho \Rightarrow (ac, bd) \in \rho$  and a compatible equivalency relation is called a congruence. If  $\rho$  is a congruence on a semigroup  $S$ , then  $S/\rho$  is a semigroup with respect to a binary operation i.e.  $(x)\rho.(y)\rho = (xy)\rho$ , for all  $x, y \in S$ . A map  $\rho^t: S \rightarrow S/\rho$  which is defined by  $(x)\rho^t = (x)\rho$  is a homomorphism. A set  $\ker \phi = \{(a, b) \in S \times S \mid (a)\phi = (b)\phi\}$  is a congruence on  $S$ .

A semigroup homomorphism  $\phi: S \rightarrow T$  has several characteristics as follow:

- a. A semigroup homomorphism  $\alpha: S/\ker \phi \rightarrow T$  is a monomorphism, such that  $R(\alpha) = R(\phi)$ .

- b. If a set  $\rho \subset \ker \phi$  is a congruence on semigroup  $S$ , then there is a unique homomorphism  $\mathcal{G}: S/\rho \rightarrow T$  such that  $R(\mathcal{G}) = R(\phi)$ .

### 3. Results:

Based on the previous researches and based on the characteristics of semigroup homomorphism, we obtain a result as follow:

**Theorem 3.** *If  $\rho \subset \ker \phi$  be a congruence on a regular  $S$ , then for every  $(a)\rho \in E(S/\rho)$  there is  $e \in E(S)$  such that  $(a)\rho = (e)\rho$ .*

**Proof:**

If  $a \in S$  is an idempotent, since  $S$  is a regular semigroup, then  $a$  is regular. But  $a = a^2$ , so there is  $a' \in S$  such that  $a^2 a' a^2 = a^2$ . Suppose  $x = a' a^2 a' \in S$ , so  $x$  is an inverse of  $a^2$ , i.e.:

$xa^2x = a'a^2a'a^2a'a^2 = a'a^2a'a^2a' = a'a^2a' = x$  and  $a^2xa^2 = a^2a'a^2a'a^2 = a^2a'a^2 = a^2$ . If  $e = axa$ , then  $e^2 = axaaxa = axa^2xa = axa = e$ . So the element  $e$  is idempotent in  $S$ . Next, we will proof that  $(a)\rho = (e)\rho$  or  $a = e \pmod{\rho}$ :

$$\begin{aligned} e = axa &\equiv a^2xa^2 \pmod{\rho} \\ &= a^2 \equiv a \pmod{\rho} \end{aligned}$$

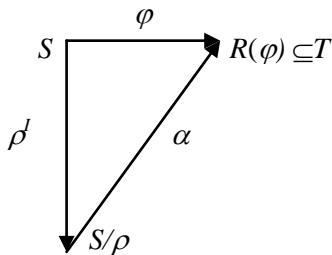
We conclude that  $(a)\rho = (e)\rho$ . ■

The second property is given as follow:

**Theorem 4.** *Let  $S$  be a regular semigroup and  $T$  be a semigroup (not need regular). If a map  $\phi$  is a homomorphism from  $S$  into  $T$ , then for every  $f \in E(R(\phi))$  there is  $e \in E(S)$  such that  $(e)\phi = f$ .*

**Proof:**

Before proving this theorem, we illustrate a commutative diagram as follow:



The maps  $\varphi, \alpha, \rho^1$  are semigroup homomorphism, which are defined as follow :

$$a \in S \longrightarrow (a)\varphi \in R(\varphi) \subseteq T$$

$$a \in S \longrightarrow (a)\rho^1 = (a)\rho$$

$$(a)\rho \in S/\rho \longrightarrow ((a)\rho)\alpha = (a)\varphi$$

The homomorphism  $\alpha$  is injective with  $R(\alpha) = R(\varphi)$ .

If  $f \in E(R(\varphi))$  is idempotent, then  $f$  is regular, since  $S$  is a regular semigroup. We have  $\alpha$  is injective, so there is a unique  $(e)\rho \in S/\rho$  such that  $((e)\rho)\alpha = f$ . The element  $(e)\rho$  is idempotent, since: If  $f \in E(R(\varphi))$ , then there is a unique  $(e)\rho \in S/\rho$  such that  $((e)\rho)\alpha = f$ . So, we obtain  $((e)\rho)\alpha = f = f^2 = ((e)\rho)\alpha$ .  $((e)\rho)\alpha = ((e)\rho)(e)\rho\alpha$ , such that  $(e)\rho = (e)\rho(e)\rho$ . By the definition, then  $((e)\rho)\alpha = (e)\varphi$ . We conclude that  $(e)\varphi = f$ .

■

We will generalize Theorem 3 and Theorem 4 i.e. we will substitute the semigroup by an arbitrary semigroup. We can do it since the commutative diagram on the proof of the previous theorem is true for an arbitrary semigroup. The sufficient condition is given in the theorem as below :

**Theorem 5.** *If  $\rho \subset \ker \phi$  is a congruence on a semigroup  $S$ , then for every idempotent regular element  $(a)\rho \in S/\rho$  there is  $e \in E(S)$  such that  $(a)\rho = (e)\rho$ .*

**Proof:**

If  $a \in S$  is an idempotent regular element, then  $(a)\rho$  is an idempotent regular, since:

There is  $a' \in S$  such that  $aa'a = a$ , so  $(a)\rho = (aa'a)\rho = (a)\rho(a')\rho(a)\rho$ . Hence there is  $(a')\rho \in S/\rho$  such that  $(a)\rho = (aa'a)\rho = (a)\rho(a')\rho(a)\rho$ . So that  $(a)\rho$  is regular and idempotent in  $S/\rho$ , since:  $(a)\rho = (a^2)\rho = (a)\rho(a)\rho$ .

The next,  $a^2$  is in  $S$  and  $a^2 = a$ , since  $a$  is idempotent. Is  $a^2$  regular?

We have  $a^{2s}a'a^2 = aa'a = a = a^2$ . Hence,  $a^2$  is idempotent in  $S$ .

Suppose  $x = a'a^2a' \in S$ , so  $x$  is an inverse of  $a^2$ , since :

$xa^2x = a'a^2a'a^2a'a^2a' = a'a^2a'a^2a' = a'a^2a' = x$  dan  $a^2xa^2 = a^2a'a^2a'a^2 = a^2a'a^2 = a^2$ . Again, suppose  $e = axa$  and we obtain  $e^2 = axaaxa = axa^2xa = axa = e$ . So that  $e$  is idempotent in  $S$ . The next, proved that  $(a)\rho = (e)\rho$ . To prove it is equivalent to prove that  $a = e \pmod{\rho}$ :

$$\begin{aligned}
e &= axa \equiv a^2 x a^2 \pmod{\rho} \\
&= a^2 \equiv a \pmod{\rho}
\end{aligned}$$

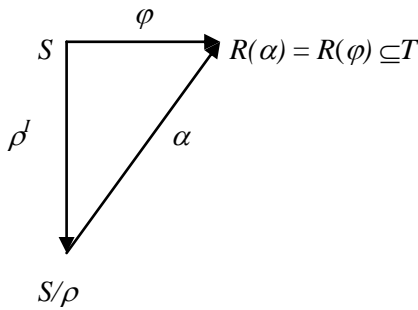
So,  $(a)\rho = (e)\rho$ . ■

As a corollary, we have the following result as a side effect of Theorem 5 and the property of semigroup homomorphism :

**Corollary 6.** *Let  $S, T$  be semigroups. If a map  $\varphi : S \rightarrow T$  is a homomorphism, then for every idempotent regular semigroup  $f \in E(R(\varphi))$  there is  $e \in E(S)$  such that  $(e)\varphi = f$ .*

**Proof:**

See the commutative diagram as below :



If  $f \in E(R(\varphi))$  is regular, then there is a unique  $(e)\rho \in (S/\rho)$  such that  $((e)\rho)\alpha = f$ . We know  $f$  is regular, so there is  $f' \in (R(\varphi))$  such that  $ff'f = f$ . On the other hand  $f' \in (R(\varphi))$ , so there is a unique  $(e')\rho \in (S/\rho)$  such that  $((e')\rho)\alpha = f'$ . But, we have  $((e)\rho(e')\rho(e)\rho)\alpha = ((e)\rho)\alpha ((e')\rho)\alpha ((e)\rho)\alpha = ff'f = f = ((e)\rho)\alpha$ , so that  $(e)\rho$  is regular.

We know that  $f^2 = f$ , so  $((e)\rho(e)\rho)\alpha = ((e)\rho)\alpha$  or  $((e^2)\rho)\alpha = ((e)\rho)\alpha$  and we obtain  $(e^2)\varphi = (e)\varphi$  or  $e^2 = e$ . Hence  $e$  is idempotent. By the definition  $((e)\rho)\alpha = (e)\varphi = f$  ■

#### 4. Conclusion

We obtain the generalization of the result of the previous research , i.e.: The idempotent elements of Bilinear form Semigrups have the same characteristics with the idempotent elements of regular semigrups and regular idempotent elements of semigroups.

## References

- [1] Howie, J. M, *An Introduction to Semigroup Theory*. Academic Press, Ltd. London, 1976.
- [2] Karyati, *Semigrup yang Dikonstruksikan dari Bentuk Bilinear*. Thesis Post Graduate Program: Gadjah Mada University , Yogyakarta, 2003.
- [3] Karyati, *Sifat Homomorphisma Semigrup Matriks Rees Atas Semigrup Bentuk Bilinear*. An Article of The Regional Conference of Indonesian Mathematics Society IX Central Java and Special District of Yogyakarta , Mathematics Department, Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Surakarta. 2003.
- [4] Karyati, *Sifat Elemen Idempoten Semigrup Bentuk Bilinear*. An Article of National Seminar, Faculty of Mathematics and Natural Sciences, State University of Yogyakarta. 2003.

***Note:***

***This paper will be submit to Southeast Asian Bulletin***