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THE FUZZY REGULARITY OF BILINEAR FORM SEMIGROUPS

1ST KARYATI, 2ND SRI WAHYUNI, 3RD BUDI SURODJO, 4TH SETIADJI

Abstract. A bilinear form semigroup is a semigroup constructed based on a bilinear form. In this paper, it is established the properties of a bilinear form semigroup as a regular semigroup. The bilinear form semigroup will be denoted by $S(B)$. We get some properties of the fuzzy regular bilinear form subsemigroup of $S(B)$, i.e: For every μ is a fuzzy regular bilinear form subsemigroup of $S(B)$ if and only if $\forall t \in (0,1]$, μ_t is a regular subsemigroup of $S(B)$, provided $\mu_t \neq \emptyset$. If A is a nonempty subset of $S(B)$, then A is a regular subsemigroup of $S(B)$ if and only if C_A, the characteristics function of A, is a fuzzy regular bilinear form subsemigroup of $S(B)$. If μ is a fuzzy regular bilinear form subsemigroup of $S(B)$, then $\mu \circ \mu = \mu$.

Keywords and Phrases : fuzzy subsemigroup, fuzzy regular bilinear form subsemigroup

1. INTRODUCTION

The basic concept of fuzzy subset is established by Zadeh since 1965 (Aktas; 2004). The concept has been developed in many areas of algebra structures, such as fuzzy subgroup , fuzzy subring, fuzzy subsemigroup, etc. Refers to the articles of Asaad (1991), Kandasamy (2003), Mordeson & Malik (1998), Ajmal (1994), Shabir (2005) , we define the fuzzy subset μ of a set S that is a mapping from S into the interval $[0,1]$, i.e. $\mu: S \to [0,1]$. Let μ be a fuzzy subset of S and $t \in [0,1]$, then the level subset μ_t is defined as set of all elements in S such that their images of the fuzzy subset μ are more than equal to t, i.e. $\mu_t = \{x \in S | \mu(x) \geq t\}$. Moreover, let S be a semigroup, then a mapping $\mu: S \to [0,1]$ is called a fuzzy subsemigroup of a semigroup S if

$$(\forall x, y \in S) \quad \mu(xy) \geq min\{\mu(x), \mu(y)\} \tag{1}$$

The property of a level subset that μ is a fuzzy subsemigroup of a semigroup S if and only if for every nonempty level subset μ_t is a subsemigroup of S.

Let X and Y be vector spaces over field K, with the characteristics of K is zero. A function $B: X \times Y \to K$ is called a bilinear form if B is linear with respect to each variable. Every bilinear form determines two linear transformation, $B_*: X \to Y^*$, which is defined as $B_*(x) = B(x, -)$ and $B^*: Y \to X^*$, which is defined as $B^*(y) = B(-, y)$. In this case, X^* and Y^* are the dual spaces of X and Y, respectively. In this paper, we denote $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ as a set of all linear operator of X and Y, respectively. For $f \in \mathcal{L}(X)$, we can construct the following sets:

$$N(f) = \{u \in X \,| f(u) = 0\}$$

$$R(f) = \{v \in X | f(x) = v, \text{ for any } x \in X\}$$

The element $(f,g) \in \mathcal{L}(X) \times \mathcal{L}(Y)$ is called an adjoin pair with respect to the bilinear form B if $B(x, g(y)) = B(f(x), y)$ for all $x \in X$ and $y \in Y$. Further, we denote the following sets:

$$\mathcal{L}'(X) = \{f \in \mathcal{L}(X) \big| N(B_*) \subseteq N(f),\ R(f) \cap N(B_*) = \{0\}\}$$

$$\mathcal{L}'(Y) = \{g \in \mathcal{L}(Y) \big| N(B^*) \subseteq N(g),\ R(g) \cap N(B^*) = \{0\}\}$$

Based on these sets, we denote a set as follow:

$$S(B) = \{(f,g) \in \mathcal{L}'(X) \times \mathcal{L}'(Y)^{op} \big| (f,g) \text{ an adjoin pair}\}$$

The structure of the set $S(B)$ is a semigroup with respect to the binary operation define as $(f,g)(f',g') = (ff', g'g)$. The semigroup $S(B)$ is called a bilinear form semigroup related to the bilinear form B.

Let S be a semigroup and $a \in S$, an element a is called a regular element if there exists $a' \in S$ such that $a = aa'a$. A semigroup S is called regular if every element is regular. The element a is called a completely regular if there exists $a' \in S$ such that $a = aa'a$ and $aa' = a'a$. A semigroup S is called a completely regular if every element is a completely regular element.

The purpose of this paper is to define the fuzzy regular bilinear form subsemigroup and investigates the characteristics of it.

2. RESULTS

Let $S(B)$, $S(B')$ be bilinear form semigroups, with respect to the bilinear form B and B', respectively. For any $(f,g) \in S(B)$, we define a set as follow:

$$R_{(f,g)} = \{(f',g') \in S(B) \big| (f,g) = (f,g)(f',g')(f,g)\}$$

and

$$C_{(f,g)} = \{(u,v) \in S(B) | (u,v)(f,g) = (f,g)(u,v)\}$$

We define a fuzzy regular bilinear form subsemigroup of $S(B)$, as follow:

Definition 2.1. *If μ is a fuzzy subsemigroup of $S(B)$ and every $(f,g) \in S(B)$, there exists $(f',g') \in R_{(f,g)}$ such that:*

$$\mu((f',g')) \geq \mu((f,g)) \qquad (2)$$

then μ is called a fuzzy regular bilinear form subsemigroup.

Based on the property of a fuzzy subsemigroup, we can investigate the following proposition:

Proposition 2.1. *The fuzzy subsemigroup μ is a fuzzy regular bilinear form subsemigroup of $S(B)$ if and only if $\forall t \in (0,1]$, μ_t is a regular subsemigroup of bilinear form semigroup $S(B)$.*

*Proof***:** It is always satisfy that μ is a fuzzy subsemigroup S if and only if μ_t is a subsemigroup of S. In fact, if μ is a fuzzy regular bilinear form subsemigroup of $S(B)$ for every $(f,g) \in S(B)$, then there exists $(f^{'},g^{'}) \in R_{(f,g)}$ such that (2) is hold, i.e. :

$$\mu(f',g') \geq \mu(f,g).$$

Furthermore,

$$\forall (f,g) \in \mu_t,\ \mu(f',g') \geq \mu(f,g) \geq t.$$

Hence, we obtain $(f^{'},g^{'}) \in \mu_t$ i.e. μ_t is a regular subsemigroup of bilinear form semigroup $S(B)$.

Conversely, suppose $\forall t \in (0,1]$,the level subset μ_t is a regular subsemigroup of bilinear form semigroup $S(B)$, provided $\mu_t \neq \emptyset$. On the other hand there is $(f,g) \in S(B)$ such that $\mu(f,g) \neq 0$ and $\forall (f^{'},g^{'}) \in R_{(f,g)}$, $\mu(f^{'},g^{'}) < \mu(f,g)$. Set $t = \mu(f,g)$. Clearly that $(f,g) \in \mu_t$ and for every $(f^{'},g^{'}) \in R_{(f,g)}$, $(f^{'},g^{'}) \notin \mu_t$. This contradicts the fact that μ_t is a regular subsemigroup of bilinear form semigroup $S(B)$. So, the equation (2) is hold.

□

The following proposition give the relation between the nonempty subset of bilinear form semigroup $S(B)$ and the characteristic function of this subset.

Proposition 2.2. *If A is a nonempty subset of $S(B)$, then A is a regular subsemigroup of bilinear form semigroup $S(B)$ if and only if C_A the characteristic function of A, is a fuzzy regular bilinear form subsemigoup of $S(B)$.*

Proof: If A is a regular subsemigroup of bilinear form semigroup $S(B)$, then the following equation is hold

$$C_A\big((f,g)(f',g')\big) = \min\{C_A\big((f,g)\big), C_A\big((f',g')\big)\}$$

and if $(f,g),(f',g') \in A$ implies $(f,g)(f',g') \in A$. For every $(f,g) \in A$, if $C_A((f,g)) \neq 0$ i.e. $C_A((f,g)) = 1$ then $(f,g) \in A$. Furthermore, since A is a regular subsemigroup of bilinear form semigroup $S(B)$, then for every $(f,g) \in A$ there exists $(f',g') \in R_{(f,g)}$ such that $(f',g') \in A$ or $C_A\big(f^{'},g^{'}\big) = 1$. Therefore,

$$C_A((f^{'},g^{'})) \geq C_A((f,g))$$

From this point we have C_A is a fuzzy regular bilinear form subsemigroup of semigroup $S(B)$.

Conversely, if C_A is a fuzzy regular bilinear form subsemigroup of $S(B)$, then for every $(f,g),(f',g') \in A$, we get

$$C_A((f,g)) = C_A((f',g')) = 1.$$

Hence,

$$C_A\big((f,g)(f',g')\big) = \min\{C_A((f,g)), C_A((f',g')\} = 1.$$

Thus,

$$C_A((f,g)(f',g')) = 1$$

and implies $(f,g)(f',g') \in A$.

In addition, if $C_A((f',g')) \geq C_A((f,g)) = 1$ then we obtain $C_A\big((f',g')\big) = 1$. Finally we conclude that $(f',g') \in A$. Thus A is a regular subsemigrup bilinear form $S(B)$. □

The following proposition give the condition of a semigroup homomorphism, such that the image of a fuzzy regular bilinear form of $S(B)$ is a fuzzy regular bilinear form subsemigroup of $S(B')$ too, and vise versa.

Proposition 2.3. *Let α be a semigroup surjective homomorphism from $S(B)$ onto $S(B')$.*

1. *If μ is fuzzy regular bilinear form subsemigroup of $S(B)$, then $\alpha(\mu)$ is fuzzy regular bilinear form subsemigroup of $S(B')$*
2. *If φ is a fuzzy regular bilinear form subsemigroup of $S(B')$, then $\alpha^{-1}(\varphi)$ is a fuzzy regular bilinear form subsemigroup of $S(B)$*

Proof:

1. For every $t \in (0,1]$, $(\alpha(\mu))_t$ is a regular subsemigroup bilinear form $S(B')$, provided $(\alpha(\mu))_t \neq \emptyset$. In the fact, if there is $t_0 \in (0,1]$ such that $(\alpha(\mu))_{t_0}$ is not a nonempty set and is not regular semigroup, then there is $(f,g) \in (\alpha(\mu))_{t_0}$ such that:

   $\forall (f',g') \in R_{(f,g)}$, $(f',g') \notin (\alpha(\mu))_{t_0}$,i.e. $\big(\alpha(\mu)\big)((f,g)) \geq t_0 \neq 0$

   and

   $$\forall (f',g') \in R_{(f,g)}\,,\ \big(\alpha(\mu)\big)((f',g')) < t_0\ ,$$

   $$\max_{(u,v)\in\alpha^{-1}(f',g')} \{\mu(u,v)\} \geq t_0$$

   and

   $$\max_{(u',v')\in\alpha^{-1}(f',g')} \{\mu(u',v')\} < t_0$$

   Let $0 < t < t_0$:

   $$\max_{(f,g)\in\alpha^{-1}(f,g)} \{\mu(u,v)\} \geq t$$

   and

   $$\max_{(u',v')\in\alpha^{-1}(f',g')} \{\mu(u',v')\} < t$$

   Now there is $(u_0,v_0) \in S(B)$ with $\alpha\big((u_0,v_{0)})\big) = (f,g)$ and $\mu((u_0,v_0) > t$ i.e. $(u_0,v_0) \in \mu_t$, $\mu_t \neq \emptyset$.

   For any $(u',v'),(f',g')$ with $\alpha((u',v') = (f',g')$, we have

   $$\mu((u',v')) < t \text{ or } (u',v') \notin \mu_t$$

Clearly, $\forall(u_0', v_0') \in R_{(u_0,v_0)}$, $\alpha(u_0', v_0') = (f', g')$ and so $(u, v) \in R_{(u_0,v_0)}$, $(u_0', v_0') \notin \mu_t$ i.e. μ_t is not regular. This is contradicts that μ is a fuzzy regular bilinear form subsemigroup of semigroup $S(B)$

2. If φ is a fuzzy regular bilinear form subsemigroup of semigroup $S(B')$, then for every $(u, v) \in S(B')$, $\mu((u, v)) \neq 0$, there exists $(u', v') \in R_{(u,v)}$ such that $\mu(u', v') \geq \mu(u, v)$. Since α is surjective, for every $(u, v) \in S(B)$ there exists $(f, g) \in S(B)$ such that $\alpha((f, g)) = (u, v)$.
For every $(f, g) \in S(B)$, $\varphi(\alpha(x)) \neq 0$, there exists

$$(\alpha((f, g))' \in R_{\alpha((f,g))}$$

with

$$\varphi((\alpha((f, g)))') \geq \varphi(\alpha((f, g)))$$

Based on the previous property, we can conclude that for every $((f, g))' \in R_{\alpha((f,g))}$, there exists $(f', g') \in R_{(f,g)}$ such that $\left(\alpha((f, g)\right)' = \alpha((f', g'))$.
Furthermore, we have:

$$\varphi((\alpha((f, g)))') \geq \varphi(\alpha((f, g))$$

$$\varphi(\alpha\left((f', g')\right) \geq \varphi(\alpha((f, g)))$$

$$\left(\alpha^{-1}(\varphi)\right)((f', g')) \geq \left(\alpha^{-1}(\varphi)\right)((f, g))$$

Finally, based on (2), $\alpha^{-1}(\varphi)$ is a fuzzy regular bilinear form subsemigroup of semigroup $S(B)$. □

Proposition 2.4. *If μ is a fuzzy regular bilinear form subsemigroup of semigroup $S(B)$, then $\mu \circ \mu = \mu$*

Proof: It is always hold $\mu \circ \mu \subseteq \mu$. For every $(f, g) \in S(B)$, if $\mu((f, g) = 0$ then

$$(\mu \circ \mu)((f, g)) \leq \mu((f, g))$$

that implies

$$(\mu \circ \mu)((f, g)) = \mu((f, g))$$

If $\mu((f, g)) \neq 0$, then there exists $(f', g') \in R_{(f,g)}$. Since μ is a fuzzy regular bilinear form subsemigrup, so we have $\mu((f', g')) \geq \mu((f, g))$. Hence

$$\max_{(u,v)(r,s)=(f,g)} \{\min\{\mu(u, v), \mu(r, s)\}\} \geq \min \left\{\mu\left((f, g)(f', g')\right), \mu\left((f, g)\right)\right\}$$

$$\geq \min \left\{\mu\left(f', g'\right), \mu(f, g)\right\}$$

It is proved that $\mu \subseteq \mu \circ \mu$, so $\mu \circ \mu = \mu$. □

3. CONCLUDING REMARK

Based on the discussion, by defining a fuzzy regular bilinear form subsemigroup, we have investigate and prove the following results.

1. The fuzzy semigroup μ is a fuzzy regular bilinear form subsemigroup of $S(B)$ if and only if $\forall t \in (0,1]$, μ_t is a regular subsemigroup of bilinear form semigroup $S(B)$

2. If A is a nonempty subset of $S(B)$, then A is a regular subsemigroup of bilinear form semigroup $S(B)$ if and only if C_A, the characteristic function of A, is a fuzzy regular bilinear form subsemigroup of $S(B)$

3. Let α be a semigroup surjective homomorphism from $S(B)$ onto $S(B')$
   a. If μ is fuzzy regular bilinear form subsemigroup of $S(B)$, then $\alpha(\mu)$ is fuzzy regular bilinear form subsemigroup of $S\left(B'\right)$
   b. If φ is a fuzzy regular bilinear form subsemigroup of $S(B')$, then $\alpha^{-1}(\varphi)$ is a fuzzy regular bilinear form subsemigroup of $S(B)$

4. If μ is a fuzzy regular bilinear form subsemigroup of $S(B)$, then $\mu \circ \mu = \mu$

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KARYATI: Ph. D student, Mathematics Department, Gadjah Mada University
E-mails: yatiuny@yahoo.com

SRI WAHYUNI : Mathematics Department, Gadjah Mada University
E-mails: swahyuni@ugm.ac.id

BUDI SURODJO: Mathematics Department, Gadjah Mada University
E-mails: surodjo_b@ugm.ac.id

SETIADJI: Mathematics Department, Gadjah Mada University