# Survey of Linear Transformation Semigroups Whose The Quasi-ideals are Bi-ideals

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## *Abstract*

*A sub semigroup $Q$ of a semigroup $S$ is called a quasi-ideal of $S$ if $SQ \cap QS \subseteq Q$. A sub semigroup $B$ of a semigroup $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$. For nonempty subset $A$ of a semigroup $S$, $(A)_q$ and $(A)_b$ denote respectively the quasi-ideal and bi-ideals of $S$ generated by $A$. Let $BQ$ denote the class of all semigroup whose bi-ideals are quasi-ideals. Let $M_D$ be a module over a division ring $D$ and $Hom(M_D)$ be a semigroup under composition of all homomorphisms $\alpha : M_D \to M_D$. The semigroup $Hom(M_D)$ is a regular semigroup.*

*In this paper we will survey which sub semigroups of $Hom(M_D)$ whose the quasi-ideals are bi-ideals.*

**Keywords** : *Semigroup, $BQ$, quasi-ideal, bi-ideal*

## I. Introduction

The notion of quasi-ideal for semigroup was introduced by O. Steinfeld, 4), in 1956. Although bi-ideals are a generalization of quasi-ideals, the notion of bi-ideal was introduced earlier by R>A Good and D.R Hughes in 1952.

A sub semigroup $Q$ of a semigroup $S$ is called a quasi-ideal of $S$ if $SQ \cap QS \subseteq Q$. A sub semigroup $B$ of a semigroup $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$. Then Quasi-ideals are a neralization of left ideals and right ideals and bi-ideals are a generalization of quasi-ideals.

O. Steinfeld has defined a bi-ideal and quasi-ideal as follows: For nonempty subset $A$ of a semigroup $S$, the quasi-ideal $(A)_q$ of $S$ generated by $A$ is the intersection of all quasi-ideal of $S$ containing $A$ and bi-ideal $(A)_b$ of $S$ generated by $A$ is the intersection of all bi-ideal of $S$ containing $A$ .1).

We use the symbol $S^1$ to denote a semigroup $S$ with an identity, otherwise, a semigroup $S$ with an identity 1 adjoined . 4). A.H. Clifford and G.H. Preston in 1) have proved that

**Proposition 1.1.** ([1], page 133) For a nonempty subset $A$ of a semigroup $S$,

$$(A)_q = S^1 A \cap AS^1 = (SA \cap AS) \cup A$$

**Proposition 1.2.** ([1], page 133) For a nonempty subset $A$ of a semigroup $S$,

$$(A)_b = \left(AS^1A\right) \cup A = (ASA) \cup A \cup A^2$$

By these definitions, $(A)_q$ and $(A)_b$ are the smallest quasi-ideal and bi-ideal, respectively, of $S$ containing $A$. Since every quasi- ideal of $S$ is a bi-ideal, it follows that for a nonempty subset $A$ of $S$, $(A)_b \subseteq (A)_q$. Hence, if $(A)_b$ is a quasi-ideal of $S$, then $(A)_b = (A)_q$, then we have:

**Proposition 1.3.** ([1], page 134) If $A$ is a nonempty subset of a semigroup $S$ such that $(A)_b \neq (A)_q$, then $(A)_b$ is a bi-ideal of $S$ which is not a quasi-ideal.

S Lajos has defined a $BQ$, that is the class of all semigroup whose bi-ideals are quasi-ideals. He has proved that:

**Proposition 1.4.** ([5], page 238) Every regular semigroup is a $BQ$-semigroup.

The next proposition is given by K.M Kapp in 5).

**Proposition 1.5.** ([5], page 238) Every left [right] simple semigroup and every left [ right] 0-simple semigroup is a $BQ$-semigroup.

In fact, J.Calais has characterized $BQ$-semigroups in 5) as follows

**Proposition 1.6.** ([5], page 238) A semigroup $S$ is a $BQ$-semigroup if and only if $(x,y)_q=(x,y)_b$ for all $x,y\in S$

Let $M_D$ be a module over a division ring $D$ and $Hom(M_D)$ be a semigroup under composition of all homomorphisms $\alpha: M_D\to M_D$. The semigroup $Hom(M_D)$ is a regular semigroup. 6).

## II. Main Results

Let $Is(M_D)$ be a set of all bijective homomorphisms of $Hom(M_D)$ , i.e.:

$$Is(M_D)=\{\alpha\in Hom(M_D)\mid \alpha \text{ is an isomorfisma}\}$$

The $Is(M_D)$ is a regular semigroup, because for all $\alpha\in Is(M_D)$, there is an $\alpha'=\alpha^{-1}\in Is(M_D)$ such that $\alpha\circ\alpha'\circ\alpha=\alpha\circ\alpha^{-1}\circ\alpha=\alpha$. By **Proposition 1.4.** the semigroup $Is(M_D)$ is in $BQ$.

The next, we construct some subsets of $Hom(M_D)$ as follow:

$$In(M_D)=\{\alpha\in Hom(M_D)\mid \alpha \text{ is injective}\}$$

$$Sur(M_D)=\{\alpha\in Hom(M_D)\mid Ran\alpha=V\}$$

$$OSur(M_D)=\{\alpha\in Hom(M_D)\mid \dim(V\setminus Ran\alpha) \text{ is infinite}\}$$

$$OIn(M_D)=\{\alpha\in Hom(M_D)\mid \dim \mathrm{Ker}\alpha \text{ is infinite}\}$$

$$BHom(M_D)=\left\{\alpha\in Hom(M_D)\,\middle|\, \begin{array}{l}\alpha \text{ is injective,}\\ \dim(V\setminus \mathrm{Ran}\alpha) \text{ is infinite}\end{array}\right\}$$

By these definitions, so we get: $Is(M_D)\subset In(M_D)$ , $Is(M_D)\subset Sur(M_D)$. The $In(M_D)$ and $Sur(M_D)$ are sub semigroups of $Hom(M_D)$:

If $\alpha,\beta\in In(M_D)$, then $Ker\alpha=\{0\}$, $Ker\beta=\{0\}$. From this condition, we have , such that $In(M_D)$ is a sub semigroup of $Hom(M_D)$

If $\alpha,\beta\in Sur(M_D)$,then $Ran\alpha=V$ , $Ran\beta=V$. From these conditions, we have $V=Ran(\alpha\circ\beta)$, such that the $Sur(M_D)$ is a sub semigroup of $Hom(M_D)$.

The $OSur(M_D)$ is a semigroup of $Hom(M_D)$, it is caused by:

If $\alpha,\beta\in OSur(M_D)$, so $\dim(V\setminus Ran\alpha)$ and $\dim(V\setminus Ran\beta)$ are infinite. For $x\in Ran(\alpha\circ\beta)$, there is $y\in V$ such that $(y)(\alpha\circ\beta)=x$ or $((y)\alpha)\beta=x$. So, $x\in Ran\beta$ and we conclude that $Ran(\alpha\circ\beta)\subseteq Ran\beta$, so $\dim(V\setminus Ran(\alpha\circ\beta))$ is infinite.

The $BHom(M_D)$ is a sub semigroup of $Hom(M_D)$:

If $\alpha,\beta\in BHom(M_D)$, then $\alpha,\beta$ are injective and the $\dim(V\setminus Ran\alpha)$ and $\dim(V\setminus Ran\beta)$ are infinite. By the previous proofing, so $\dim(V\setminus Ran(\alpha\circ\beta))$ are infinite and $(\alpha\circ\beta)$ is bijective

In order to prove that . $In(M_D)$ and $Sur(M_D)$ are in $BQ$ if and only if $\dim V$ is finite, we need this Lemma:

**Lemma 2.1.** ([2], page 407) If $B$ is a basis of $M_D$, $A\subseteq B$ and $\alpha\in Hom(M_D)$ is one-to-one, then

$$\dim(Ran\alpha/\langle A\rangle\alpha)=|B\setminus A|$$

From this Lemma, so we can prove that:

**Theorem 2.2.** ([2], page 408) $In(M_D)\in BQ$ if and only if $\dim M_D$ is finite.

Proof:

If $\dim M_D$ is finite, then $In(M_D)=Is(M_D)$. So, $In(M_D)$ is a regular semigroup. By **Proposition 1.4.** $In(M_D)\in BQ$.

The other side, assume that $\dim M_D$ is infinite. Let $B$ be a basis of $M_D$, so $|B|$ is infinite. Let $A=\{u_n\mid n\in N\}$ is a subset of $B$, where for any distinct $i,j\in N$, $u_i\neq u_j$.

Let $\alpha,\beta,\gamma\in Hom(M_D)$ be defined as follow:

$$(v)\alpha=\begin{cases}u_{2n} & \text{if } v=u_n \text{ for some } n\in N\\ v & \text{if } v\in B\setminus A\end{cases}$$

$$(v)\beta=\begin{cases}u_{n+1} & \text{if } v=u_n \text{ for some } n\in N\\ v & \text{if } v\in B\setminus A\end{cases}$$

$$(v)\gamma=\begin{cases}u_{n+2} & \text{if } v=u_n \text{ for some } n\in N\\ v & \text{if } v\in B\setminus A\end{cases}$$

By this definition, so $\ker\alpha=\ker\beta=\ker\gamma=\{0\}$, such that $\alpha,\beta,\gamma\in In(M_D)$. Next, we have $(u_n)(\beta\circ\alpha)=(u_n)(\alpha\bullet\gamma)$, for all $n\in N$ and for all $v\in B\setminus A$, we have $(v)(\beta\circ\alpha)=((v)\beta)\alpha=(v)\alpha=v$

and $(v)(\alpha \circ \gamma) = ((v)\alpha)\gamma = (v)\gamma = v$. So we conclude that $\alpha \neq \beta \circ \alpha = \alpha \circ \gamma$. By this conditions we have $\beta \circ \alpha \in In(M_D)\alpha$, because $\beta \in In(M_D)$ and $\alpha \circ \gamma \in \alpha\ In(M_D)$. We know that $\beta \circ \alpha = \alpha \circ \gamma$, so by the **Proposition 1.1** we have $\beta \circ \alpha \in In(M_D)\alpha \cap \alpha In(M_D) = (\alpha)_q$. Suppose that $\beta \circ \alpha \in (\alpha)_q$, because $\alpha \neq \beta \circ \alpha$, by **Proposition 1.1** we get $\beta \circ \alpha \in \alpha\ In(M_D)\alpha$. Let $\lambda \in In(M_D)$ such that $\beta \circ \alpha = \alpha \circ \lambda \circ \alpha$. Since $\alpha$ is injective, so $\beta = \lambda \circ \alpha$. Then we have:

$$B \setminus \{u_1\} = B\beta = B(\alpha \circ \lambda) = (B\alpha)\lambda = (B \setminus \{u_{2n-1} \mid n \in N\})\lambda$$

By **Lemma 2.1**, we have:

$$\dim\left(Ran\lambda / \langle (B \setminus \{u_{2n-1} \mid n \in N\})\lambda \rangle\right) = \left|\{u_{2n-1} \mid n \in N\}\right|$$

From these conditions, so this condition is hold: $\dim\left(Ran\lambda / \langle B \setminus \{u_1\}\rangle\right) = \left|\{u_{2n-1} \mid n \in N\}\right|$, but in the other hand :

$$\dim\left(Ran\lambda / \langle B \setminus \{u_1\}\rangle\right) \leq \dim\left(V / \langle B \setminus \{u_1\}\rangle\right) = 1.$$

So, there is a contradiction. By Proposition 1.3, we have $\beta \circ \alpha \notin (\alpha)_b$, then $In(M_D) \notin BQ$.

**Theorem 2.3**. ([2], page 408) $Sur(M_D) \in BQ$ if and only if $\dim Hom(M_D)$ is finite

Proof:

The proof of this theorem is similar with the previous theorem.

The other semigroups e.i. $OSur(M_D)$ a always belongs to $BQ$ but it is not regular and is neither right 0-simple nor left 0-simple, if $\dim M_D$ is finite. These condition is guarantied by these propositions:

**Propositions 2.4.** ([2], page 409) The semigroup $OSur(M_D)$ isn't regular.

**Propositions 2.5.** ([2], page 410) The semigroup $OSur(M_D)$ is neither right 0-simple nor left 0-simple.

Although $OSur(M_D)$ has properties likes above, $OSur(M_D)$ is a left ideal of $Hom(M_D)$ and is always in $BQ$.

For the other semigroup, i.e. $BHom(M_D)$ is in $BQ$ if and only if $\dim M_D$ is countably infinite. It is caused by this lemma:

**Lemma 2.6.** ([2], page 411) If $\dim M_D$ is countably infinite, then $BHom(M_D)$ is right simple.

The next, we construct the other subsets of $Hom(M_D)$:

$$OInSur(M_D) = \{\alpha \in Hom(M_D) \mid \dim Ker\alpha, \dim(V \setminus Ran\alpha) \text{ are infinite}\}$$

$$OBHom(M_D) = \{\alpha \in Hom(M_D) \mid Ran\alpha = M_D, \dim Ker\alpha \text{ is infinite}\}$$

From the definition, we get: $OInSur(M_D) = OSur(M_D) \cap OIn(M_D)$, this set is not empty set, because $0 \in OSur(M_D) \cap OIn(M_D) = OInSur(M_D)$. This set is a sub semigroup of $Hom(M_D)$.

**Lemma 2.7.** ([5], page 240) For every infinite dimention of $M_D$, $OInSur(M_D)$ is a regular sub semigroup of $Hom(M_D)$

The following theorem is the corollary of the previous lemma:

**Theorem 2.8.** ([5], page 240) For every infinite dimention of $M_D$, $OInSur(M_D)$ is in $BQ$

The set $OBHom(M_D)$ is an intersection of $In(M_D)$ and $OSur(M_D)$. Let $B$ be a basis of $M_D$, since $B$ is infinite, there is a subset $A$ of $B$ such that $|A| = |B \setminus A| = |B|$. Then there exist a bijection $\varphi : A \to B$. Define a homomorphisms in $Hom(M_D)$ as follow:

$$(v)\alpha = \begin{cases} \varphi(v) & if \quad v \in A \\ 0 & if \quad v \in B \setminus A \end{cases}$$

Hence $\alpha \in OBHom(M_D)$

**Lemma 2.9.** ([5], page 241). If $\dim M_D$ is countably infinite, then $OBHom(M_D)$ is left simple.

As the corollary, we get:

**Theorem 2.10**. ([5], page 242). The smeigroup $OBHom(M_D)$ is in $BQ$ if and only if $\dim M_D$ is countably infinite.

## III. Conclusion

From this survey, we can conclude that there is a different conditions such that the sub semigroup of $Hom(M_D)$ is in $BQ$, i.e:

1. $In(M_D) \in BQ$ if and only if $\dim M_D$ is finite.
2. $Sur(M_D) \in BQ$ if and only if $\dim Hom(M_D)$ is finite
3. The semigroup $OSur(M_D)$ isn't regular.

4. The semigroup $OSur(M_D)$ is neither right 0-simple nor left 0-simple.
5. If $\dim M_D$ is countably infinite , then $BHom(M_D)$ is right simple.
6. For every infinite dimension of $M_D$, $OInSur(M_D)$ is a regular sub semigroup of $Hom(M_D)$
7. For every infinite dimention of $M_D$, $OInSur(M_D)$ is in $BQ$
8. If $\dim M_D$ is countably infinite, then $OBHom(M_D)$ is left simple.
9. The smeigroup $OBHom(M_D)$ is in $BQ$ if and only if $\dim M_D$ is countably infinite.
10. Finally, we can conclude that not every semigroup in $BQ$ is a regular semigroup and not every semigroup in $BQ$ is either right 0-simple or left 0-simple.

## V. References


1. Baupradist, S, Kemprasit, Y,” Existence of Bi-ideals which are not Quasi-ideals in Semigroups of Continuous Functions and Semigroups of Differentiable Functions”, Proceeding of the International Conference on Algebra and Its Applications, 133,143 (2002)
2. Namnak, C, Kemprasit, Y,”On Semigroups of Linear Transformations whose Bi-ideals are Quasi-ideals”, PU.M.A Vol 12 No4, 405,413 (2001)
3. Namnak, C, Kemprasit, Y, “Some Semigroups of Linear Transformations Whose Sets of Bi-Ideals and Quasi-ideals Coincide”, Proceeding of the International Conference on Algebra and Its Applications, 215,224 (2002)
4. Namnak, C, Kemprasit, Y, “ Generalized Transformatioan Semigroups Whose Bi-ideals and Quasi-ideals Coinside”, Southeast Asian Bulletin of Mathematics Vol 27, SEAMS, 623-630, (2003)
5. Namnak, C, Kemprasit, Y, “Some BQ-Semigroups of Linear Transformations”, Kyungpook Mathematics Journal 43, 237, 246 (2003)
6. Kemprasit, Yupaporn,”Regularity and Unit-Regularity of Generalized Semigroups of Linear Transformations” , Southeast Asian Bulletin of Mathematics Vol 25, Springer-Verlag, 617-622, (2002)