

### Section 2.3. Matrix Inverses

Three basic operations on matrices, addition, multiplication, and subtraction, are analogues for matrices of the same operations for numbers. In this section we introduce the matrix analogue of numerical division.

To begin, consider the numerical equation:

$$ax=b$$

is solved when  $a$  and  $b$  are known numbers. If  $a=0$ , there is no solution (unless  $b=0$ ). But if  $a \neq 0$ , we can multiply both sides by inverse of  $a$ , denoted by  $a^{-1}$ , to obtain the solution  $x=a^{-1}b$ . This multiplication by  $a^{-1}$  is commonly called dividing by  $a$ , and the property of  $a^{-1}$  that makes the work is that  $a^{-1}a=1$ . Moreover, we saw in the previous section that the role that 1 plays in arithmetic is played in matrix algebra by the identity matrix  $I$ .

This suggests the following definition. If  $A$  is a square matrix, a matrix  $B$  is called an inverse of  $A$  if and only if:

$$AB=I \text{ and } BA=I$$

A matrix  $A$  that has an inverse is called an invertible matrix.

**Example 24** Show that  $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$  is an inverse of  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

**Solution:**

Compute  $AB$  and  $BA$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,  $AB=I=BA$ , so matrix  $B$  is an inverse of  $A$

**Example 25** Show that  $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$  has no solution

**Solution:**

Let  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  denote an arbitrary  $2 \times 2$  matrix. Then:

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+3c & b+3d \end{bmatrix}$$

so  $AB$  has a row of zeros. Hence  $AB$  cannot equal  $I$  for any  $B$ .

Example 25 shows that it is possible for a nonzero matrix to have no inverse. But if a matrix has an inverse, it has only one.

**Theorem 7** If a matrix has an inverse, it is unique

**Proof**

Let  $A$  is a matrix that has two inverse, these called  $B$  and  $C$ . Since  $B$  and  $C$  are both inverses of  $A$ , we have  $CA = I = AB$ . Hence  $B = IB = (CA)B = C(AB) = CI = C$ .

If somehow a matrix  $B$  can be found such that  $AB = I = BA$ , then  $A$  is an invertible and  $B$  is the inverse of  $A$ , in symbols,  $B = A^{-1}$ .

**Example 26** If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $ad - bc \neq 0$ , show that  $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**Solution:**

We verify that  $AA^{-1} = I$  and we leave  $A^{-1}A = I$  to the reader

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I \end{aligned}$$

**Exercises:**

1. Find the inverse of the matrices as follow. ( Use the definition of an inverse of a matrix ):

a.  $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$       b.  $\begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix}$       c.  $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

2. Find the inverse of the matrices as follow. ( Use the definition of an inverse of a matrix )

a.  $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$       b.  $\begin{bmatrix} -1 & 1 & 2 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

**Inverse and Linear System**

Matrix inverses can be used to solve certain systems of linear equations. A system of linear equations can be written as a single matrix equation:

$$AX = B$$

where  $A$  and  $B$  are known matrices and  $X$  is to be determined. If  $A$  is invertible, then multiply each side of the equation the left by  $A^{-1}$  to get:

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

This gives a solution to the system of equations, because if  $X = A^{-1}B$  is substitute to this equation will be hold:

$$AX = A A^{-1}B = IB = B$$

Furthermore, the argument shows that if  $X$  is any solution, then necessarily  $X = A^{-1}B$ , so the solution is unique. Of course the technique works only when the coefficient matrix  $A$  has an inverse.

**Theorem 8** Suppose a system of  $n$  equations in  $n$  variables is written in matrix form as:

$$AX = B$$

If the  $n \times n$  coefficient matrix  $A$  is invertible, the system has the unique solution:

$$X = A^{-1}B$$

**Example 27** Solve this system of linear equations

$$x - 2y = 3$$

$$2x + y = 0$$

**Solution:**

Be written in matrix form:

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

We get the inverse of the coefficient matrix as follow:

$$\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

So the solution of system of linear equations is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix}$$

### An Inversion Method

Given the invertible  $2 \times 2$  matrix  $A$ , we determined  $A^{-1}$  from the equation  $AA^{-1} = I$ . Write

$$A^{-1} = \begin{bmatrix} a & b \\ x & y \end{bmatrix}$$

where  $a, b, x, y$  are to be determined. Equating columns in the equation  $AA^{-1} = I$  gives

$$A \begin{bmatrix} a \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} b \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are systems of linear equations, each with  $A$  as coefficient matrix. Since  $A$  is invertible, each system has a unique solution by Theorem 2. But this means that the reduced row – echelon form  $R$  of  $A$  cannot have a row of zeros, and so is the identity matrix ( $R$  is square). Hence, there is a sequence of elementary row operations carrying  $A$  to the  $2 \times 2$  identity matrix  $I$ . This sequence carries the augmented matrices of both system to reduced row echelon form and so solves the systems:

$$\left[ A \mid I \right] \rightarrow \left[ I \mid a \right] \quad \left[ A \mid 0 \right] \rightarrow \left[ I \mid b \right]$$

Hence, we can do both calculations simultaneously.

$$\left[ A \mid I \mid 0 \right] \rightarrow \left[ I \mid a \mid b \right]$$

This can be written more compactly as follows:

$$\left[ A \mid I \right] \rightarrow \left[ I \mid A^{-1} \right]$$

In other words, the sequence of row operations that carries  $A$  to  $I$  also carries  $I$  to  $A^{-1}$ .

### Matrix Inversion Algorithm

*If  $A$  is an invertible matrix, there exists a sequence of elementary row operations that carry  $A$  to the identity matrix  $I$  of the same size, written  $A$*

$\rightarrow I$ . This same series of row operations carries  $I$  to  $A^{-1}$ , that is  $I \rightarrow A^{-1}$ . The algorithm can be summarized as follows:

$$\left[ \begin{array}{c|c} A & I \end{array} \right] \rightarrow \left[ \begin{array}{c|c} A^{-1} & \end{array} \right]$$

where the row operations on  $A$  and  $I$  are carried out simultaneously.

**Example 28** Use the inversion algorithm to find the inverse of the matrix:

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

**Solution**

Apply elementary row operations to the double matrix:

$$\left[ \begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array} \right]$$

So as to carry  $A$  to  $I$ . First interchange rows 1 and 2.

$$\left[ \begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array} \right]$$

Subtract 2 times row 1 from row 2, and subtract row 1 from row 3

$$\left[ \begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array} \right]$$

Continue to reduced row-echelon form.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 11 & 4 & 7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$\text{Hence, } A^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$$

**Theorem 9** *If  $A$  is an  $n \times n$  matrix, either  $A$  can be reduced to  $I$  by elementary row operations or it cannot. In the first case, the algorithm produces  $A^{-1}$ ; in the second case,  $A^{-1}$  does not exist.*

### Properties of Inverses

**Theorem 10** *All the following matrices are square matrices of the same size:*

1.  $I$  is invertible and  $I^{-1} = I$
2. If  $A$  is invertible, so is  $A^{-1}$ , and  $(A^{-1})^{-1} = A$
3. If  $A, B$  are invertible, so is  $AB$  and  $(AB)^{-1} = B^{-1} A^{-1}$
4. If  $A_1, A_2, \dots, A_k$  are all invertible matrices, so is the product  $A_1 A_2 \dots A_k$  and  $(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$
5. If  $A$  is invertible, so is  $A^k$  for  $k \geq 1$  and  $(A^k)^{-1} = (A^{-1})^k$
6. If  $A$  is invertible and  $a \neq 0$  is a number, then  $aA$  is invertible and  $(aA)^{-1} = \frac{1}{a} A^{-1}$
7. If  $A$  is invertible, so is its transpose  $A^T$  and  $(A^T)^{-1} = (A^{-1})^T$

**Corollary** *A square matrix  $A$  is invertible if and only if  $A^T$  is invertible.*

**Example 29** Find  $A$  if  $(A^T - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

**Solution**

By Theorem 4 (2) we obtain:

$$(A^T - 2I) = [(A^T - 2I)^{-1}]^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\text{Hence } A^T = 2I + \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}, \text{ so } A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

**Theorem 10** *The following conditions are equivalent for an  $n \times n$  matrix  $A$ :*

1.  $A$  is invertible
2. The homogeneous system  $AX = 0$  has only the trivial solution  $X = 0$
3.  $A$  can be carried to the identity matrix  $I_n$  by elementary row operations.
4. The system  $AX = B$  has at least one solution  $X$  for every choice of column  $B$ .
5. There exists an  $n \times n$  matrix  $C$  such that  $AC = I_n$

**Corollary** *If  $A$  and  $C$  are square matrices such that  $AC = I$ . In particular, both  $A$  and  $C$  are invertible,  $C = A^{-1}$  and  $A = C^{-1}$ .*

### Exercises 2.3

1. Find the inverse of each of the following matrices:

a.  $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

b.  $\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 5 & -1 \end{bmatrix}$

c.  $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 6 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 1 \end{bmatrix}$

2. In each case, solve the system of equations by finding the inverse of the coefficient matrix:



$$\begin{aligned} x + y + 2z &= 5 \\ \text{a. } x + y + z &= 0 \\ x + 2y + 4z &= -2 \end{aligned}$$

$$\begin{aligned} x + 4y + 2z &= 1 \\ \text{b. } 2x + 3y + 3z &= -1 \\ 4x + y + 4z &= 0 \end{aligned}$$

$$\begin{aligned} x + y + z + w &= 1 \\ \text{c. } x + y &= 0 \\ y + w &= -1 \\ x + w &= 2 \end{aligned}$$

3. Given  $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$

a. Solve the system of equation  $AX = \begin{bmatrix} -1 & 3 \end{bmatrix}^T$

b. Find a matrix  $B$  such that  $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

4. Find  $A$  when :

a.  $(3A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$       b.  $\left( \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$       c.  $(A^{-1} - 2I)^T = -2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

5. If  $A, B, C$  are square matrices and  $AB = I = CA$ , show that  $A$  is invertible and

$$B = C = A^{-1}$$

6. Verify that  $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$  satisfies  $A^2 - 3A + 2I = 0$  and use this fact to show that

$$A^{-1} = \frac{1}{2}(3I - A)$$

7. Let  $A, B$  denote  $n \times n$  invertible matrices:

a.  $A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}$

b. If  $A + B$  is invertible, so that  $A^{-1} + B^{-1}$  is invertible and find a formula for

$$(A^{-1} + B^{-1})^{-1}$$