

# 1

## Systems of Linear Equations

### Section 1.1 Solutions and Elementary Operations

Practical problems in many fields of study – such as biology, business, chemistry, computer science, economics, electronics, engineering and social sciences-can often be reduced to solving a system of linear equations. Linear Algebra arose from attempts to find systematic method for solving these systems. Here is the example:

**Example 1.** A biologist wants to create a diet from fish and meal that contains 183 grams of protein and 93 grams of carbohydrate per day. If fish contains 70% protein and 10% carbohydrate, and meal 30% protein and 60% carbohydrate, how much of each food is required each day?

#### **Solution**

If  $x$  grams of fish and  $y$  grams of meal are given each day, the amount of protein and carbohydrate in the diet are represented by these equations:

$$0.7x + 0.3y = 183$$

$$0.1x + 0.6y = 93$$

Multiply the second equation by 7 gives

$$0.7x + 0.3y = 183$$

$$0.7x + 4.2y = 651$$

Subtracting the top equation from the bottom equation gives  $3.9y = 468$  ; then dividing by 3.9 gives  $y = 120$ . Substituting this into the equation  $0.1x + 0.6y = 93$  gives  $x = 210$ . Hence, the diet consists of 210 grams of fish and 120 grams of meal per day.

If  $a, b, c$  are real numbers, the graph of an equation of the form

$$ax + by = c$$

is a straight line ( if  $a, b$  are not both zero ), so such an equation is called a linear equation in the variable  $x$  and  $y$ . However, it is often convenient to write the variables as  $x_1, x_2, x_3, \dots, x_n$ , particularly when more than two variables are involved. An equation of the form:

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

is called a linear equation in the  $n$  variables  $x_1, x_2, x_3, \dots, x_n$ . Here  $a_1, a_2, a_3, \dots, a_n$  denote real numbers ( called the coefficients of  $x_1, x_2, x_3, \dots, x_n$ , respectively ) and  $b$  is also a number ( called the constant term of the equation ). A finite collection of linear equations in the variables  $x_1, x_2, x_3, \dots, x_n$  is called a system of linear equations in these variables. Hence,

$$2x_1 - 3x_2 + 5x_3 = 7$$

is a linear equation; the coefficient of  $x_1, x_2, x_3$  are 2, -3, 5, respectively, and the constant term is 7. Note that each variable in a linear equation occurs to the first power only.

Given a linear equation  $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$ , a sequence  $s_1, s_2, s_3, \dots, s_n$  of  $n$  numbers is called a solution to the equation if:

$$a_1s_1 + a_2s_2 + a_3s_3 + \dots + a_ns_n = b$$

that is, if the equation is satisfied when the substitutions  $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$  are made. A sequence of numbers is called a solution to a system of equations if it is a solution to every equation the system.

## Example 2

Show that, for arbitrary values of  $s, t$ ,

$$x = t - s + 1$$

$$y = t + s + 2$$

$$z = s$$

$$w = t$$

is solution to the system :

$$x - 2y + 3z + w = -3$$

$$2x - y + 3z - w = 0$$

### Solution

Simply substitute these values of  $x, y, z, w$  in each equation.

$$x - 2y + 3z + w = (t - s + 1) - 2(t + s + 2) + 3s + t = -3$$

$$2x - y + 3z - w = 2(t - s + 1) - (t + s + 2) + 3 - t = 0$$

Because both equations are satisfied , it is a solution for all  $s, t$  .

The quantities  $s, t$  in Example 2 are called **parameters**, and the set of solutions, describe in this way, is said to be given in **parametric form** and is called the general solution to the system. It turns out that the solutions to every system of equations ( if there are solutions ) can be given in parametric form. The following example shows how this happens in the simplest systems where only one equation is present.

### Example 3

Describe all solution to  $3x - y + 2z = 6$  in parametric form.

### Solution

Solving the equation for  $y$  in terms of  $x$  and  $z$  , we get  $y = 3x + 2z - 6$  . If

$s$  and  $t$  are arbitrary then, setting  $x = s$  ,  $z = t$  , we get solution:

$$x = s$$

$$y = 3 + 2t - 6$$

$$z = t$$

Of course we could have solved for  $x$  :

$$x = \frac{1}{3}(y - 2z + 6)$$

Then, if we take  $y = p, z = q$  , the solutions are represented as follows:

$$x = \frac{1}{3}(p - 2q + 6)$$

$$y = p, z = q,$$

$p, q$  arbitrary

The same family of solutions can “look” quite different!

When only two variables are involved, the solutions to systems of linear equations can be described geometrically because the graph of a linear equation  $ax + by = c$  is a straight line if  $a$  and  $b$  are not both zero. Moreover a point  $P(s, t)$  with coordinates  $s$  and  $t$  lies on the line if and only if  $as + bt = c$ , that is when  $x = s$  and  $y = t$  is a solution to the equation. Hence, the solution to a system of linear equations correspond to the points  $P(s, t)$  that lie on all the lines in equation.

In particular, if the system consists of just one equation, there must be infinitely many solutions because there are infinitely many points on a line. If the system has two equations, there are three possibilities for the corresponding straight lines:

1. The lines intersect in a single point. Then the system has a unique solution corresponding to that point.
2. The lines are parallel ( and distinct ) and so do not intersect. Then the system has no solution.
3. The lines are identical. Then the system has infinitely many solutions, i.e. one for each point on the ( common ) line.

When three variables are present, the graph of an equation  $ax + by + cz = d$  can be shown to a plane and so again provides a “picture” of the set of solution. However, this graphical method has its limitations: when more than three variables are involved, no physical image of the graphs ( called hyperplanes ) is possible. It is necessary to turn to a more “ algebraic” method of solution.

Before describing the method, we introduce a concept that simplifies the computation involved. Consider the following system:

$$3x + 2y - z + w = -1$$

$$2x - z + 2w = 0$$

$$3x + y + 2z + 5w = 2$$

of three equations in four variables. The array of numbers:

$$\left[ \begin{array}{cccc|c} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{array} \right]$$

occurring in the system is called the **augmented matrix** of system. Each row of the matrix consist of the coefficients of the variables ( in order ) from the corresponding equation, together with the constant term. For clarity, the constants are separated by a vertical line. The augmented matrix is just a different way of describing the system of equations. The array of coefficients of the variables

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$

is called the **coefficient matrix** of the system and  $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$  is called the

**constant matrix** of the system.

## Elementary Operations

The algebraic method for solving systems of linear equations is described as follows. Two such systems are said to be equivalent if each has the same set of solutions. A system is solved by writing a series of system, one after the other, each equivalent to the pervious system. Each of these systems has the same set of solutions as the original one; the aim is to end up with a system that is easy to solve. Each system in the series is obtained from the preceding system by a simple manipulation chosen so that it does not change the set of solutions.

As an illustration, we solve the system  $x + 2y = -2$  ,  $2x + 7y = 7$  in this manner. At each stage, the corresponding augmented matrix is displayed. The original system is:

$$\begin{array}{l} x + 2y = -2 \\ 2x + y = 7 \end{array} \quad \left[ \begin{array}{cc|c} 1 & 2 & -2 \\ 2 & 1 & 7 \end{array} \right]$$

First, subtract twice the first equation from the second. The resulting system is:

$$\begin{array}{l} x + 2y = -2 \\ -3y = 11 \end{array} \quad \left[ \begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array} \right]$$

which is equivalent to the original. At this stage we obtain  $y = -\frac{11}{3}$  by multiplying the second equation by  $-\frac{1}{3}$ . The result is the equivalent system:

$$\begin{array}{l} x + 2y = -2 \\ y = -\frac{11}{3} \end{array} \quad \left[ \begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -\frac{11}{3} \end{array} \right]$$

Finally, we subtract twice the second equation from the first to get another equivalent system.

$$\begin{array}{l} x = \frac{16}{3} \\ y = -\frac{11}{3} \end{array} \quad \left[ \begin{array}{cc|c} 1 & 0 & \frac{16}{3} \\ 0 & 1 & -\frac{11}{3} \end{array} \right]$$

Now this system is easy to solve! And because it is equivalent to the original system, it provides the solution to that system.

Observe that, at each stage, a certain operation is performed on the system ( and thus on the augmented matrix ) to produce an equivalent system. The following operations, called elementary operations, can routinely be performed on systems of linear equations to produce equivalent systems.

- I. Interchange two equations
- II. Multiply one equation by a nonzero number
- III. Add a multiple of one equation to a different equation.

### **Theorem 1**

*Suppose that an elementary operation is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.*

Elementary operation performed on a system equations produce corresponding manipulations of the rows of the augmented matrix. Thus,

multiplying a row of a matrix by a number  $k$  means multiplying every entry of the row by  $k$ . Adding one row to another row means adding each entry of that row to the corresponding entry of the other row. Subtracting two rows is done similarly.

In hand calculations ( and in computer programs ) we usually manipulate the rows of the augmented matrix rather than the equations. For this reason we restate these elementary equations for matrices.

The following are called **elementary row operations** on a matrix :

- I. Interchange two rows
- II. Multiply one row by a nonzero number
- III. Add a multiple of one row to a different row.

In the illustration preceding Theorem 1 these operations led to a matrix of the form:

$$\left[ \begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array} \right]$$

where the asterisks represent numbers. In the case of three equations in the three variables, the goal is to produce a matrix of the form:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right]$$

This does not always happen, as we will see in the next section. Here is an example in which it does happen.

**Example 4** Find all solutions to the following system of equations

$$3x + 4y + z = 1$$

$$2x + 3y = 0$$

$$4x + 3y - z = -2$$

**Solution**

The augmented matrix of the original system is:

$$\left[ \begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array} \right]$$

To create a 1 in the upper left corner we could multiply row 1 through by  $\frac{1}{3}$ . However, the 1 can be obtained without fractions by subtracting row 2 from row 1. The result is:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array} \right]$$

The upper left 1 is now used to 'clean up' the first column, that is create zeros in the other positions in the column. First subtract 2 times row 1 from row 2 to obtain:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 4 & 3 & -1 & -2 \end{array} \right]$$

Next subtract 4 times row 1 from row 3. The result is:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array} \right]$$

This completes the work on column 1. We now use the 1 in the second position of the second row to clean up the second column by subtracting row 2 from row 1 and adding row 2 to row 3. For convenience, both row operations are done in one step. The result is

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{array} \right]$$

Note that these manipulations did not affect the first column( the second row has a zero there ), so our previous effort there has not been undermined. Finally we clean up the third column. Begin by multiplying row 3 by  $-\frac{1}{7}$  to obtain:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{8}{7} \end{array} \right]$$

Now subtract 3 times row 3 from row 1, and add 2 times row 3 to row 2 to get:



$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{array} \right]$$

The corresponding equations are  $x = -\frac{3}{7}, y = \frac{2}{7}, z = \frac{8}{7}$ , which give the solutions

### Exercises 1.1

1. Verify that the following is solution for all values of  $s, t$

$$x = 2s + 12t + 13$$

$$y = s$$

$$z = -s - 3t - 3$$

$$w = t$$

is a solution of

$$2x + 5y + 9z + 3w = -1$$

$$x + 2y + 4z = 1$$

2. Find all solutions to the following in all possible parametric form :

a.  $3x - 2y = 9$

c.  $-x + 2y + 3z = 7$

b.  $-2a + 2b = 5$

d.  $4\alpha - 2\beta - 3\gamma = 9$

3. Regarding  $4x - 2y = 3$  as the equation  $4x - 2y + 0z = 3$  in three variables, find all solutions in parametric form!

4. Write the augmented matrix for each of the following systems of linear equations:

a.  $x - 2y + z = 4$

$a + b = 6$

a.  $2x + y - z = 7$

b.  $b + c = 7$

$3y + 3z = 9$

$a - c = 8$

5. Write the system of linear equations that has each of the following augmented matrix:

a. 
$$\left[ \begin{array}{cccc|c} 1 & 0 & -2 & 2 & 10 \\ 2 & -1 & 1 & 1 & 6 \\ 1 & -1 & 3 & 1 & -3 \\ 2 & 2 & 1 & -1 & 0 \end{array} \right]$$

b. 
$$\left[ \begin{array}{ccc|c} -1 & 0 & 2 & 9 \\ 4 & -5 & 3 & -4 \\ 3 & -2 & 3 & 5 \end{array} \right]$$

6. A small furniture factory manufactures tables and chairs. It takes 2 hours to assemble a table and 30 minutes to assemble a chair. Assembly is carried out by 4

workers on the basis of a single 8-hours shift per day. Customers always buy at four chairs with each table. Find the model of this case as a system of linear equations.

7. A farmer owns 200 pigs that consume 90 lb of special feed daily. The feed is prepared as a mixture of corn and soybean meal with the following compositions:

Feedstuff	Pounds per pound of Feedstuff		
	Calcium	Protein	Fiber
Corn	0.001	0.09	0.02
Soybean meal	0.002	0.60	0.06

The dietary requirements of the pigs are: 1% calcium, 30% protein, 5 % fiber. Find the model of this case as a system of linear equations !

8. Two product are manufactured by passing sequentially through three machines. Times per machine allocated to the two products is 10 hours per day. The production time per unit of each product are:

Product	Minutes per unit		
	Machine 1	Machine 2	Machine 3
1	10	6	8
2	5	20	15

Find the model of this case as a system of linear equations !

9. A company produces two products, A and B. The sales volume for product A is 60% of the total sales of the two products. The two product use the same material, of which the daily availability is 100 lb. Products A and B use this raw material at the rates of 2 lb/unit and 4 lb/unit, respectively. Find the model of this case as a system of linear equations !

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