Section 2.2. Matrix Multiplication

Matrix multiplication is a little more complicated than matrix addition or scalar multiplication. If A is an $m \times n$ matrix, then B is an $n \times k$ matrix, 9 the product AB of A and B is the $m \times k$ matrix whose (i, j)- entry is computed as follow:

Multiply each entry of row i of A by the corresponding entry of column j of B, and add the result

This is called the dot product of row i of A and column j of B.

Example 20 Compute the $(1,3)$ and $(2,4)$- entries of AB where:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

Then compute AB.

Solution

The $(1,3)$- entry of AB is the dot product of row 1 of A and column 3 of B, computed by multiplying corresponding entries and adding the result:

$$3.6 + (-1).3 + 2.5 = 18 - 3 + 10 = 25$$

Similarly, the $(2,4)$ entry of AB is the dot product of row 2 of A and column 4 of B, computed by multiplying corresponding entries and adding the result:

$$0.0 + 1.4 + 4.8 = 0 + 4 + 32 = 36$$

Since A is 2×3 and B is 3×4, the product is a 2×4 matrix:

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Computing the (i, j) - entry of AB involves going across row i of A and down column j of B, multiplying corresponding entries, and adding the results. This requires that the rows of A and the columns of B be the same length. The following rule is a useful way to remember when the product of A and B can be formed and what the size of the product matrix is.

Rule

Suppose A and B have sizes $m \times n$ and $n' \times p$, respectively:

$$m \times \overbrace{n \quad n'} \times p$$

The product AB can be formed only when $n = n'$; in this case , the product matrix AB is of size $m \times p$. When this happens, the product AB is defined.

Example 21 If $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, Compute A^2, AB, BA, B^2 when they are defined.

Solution

Here, A is a 1×3 matrix and B is a 3×1 matrix, so A^2, B^2 are not defined. The AB and BA are defined, these are 1×1 and 3×3 matrices, respectively:

$$AB = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.5 + 3.6 + 2.4 \end{bmatrix} = \begin{bmatrix} 5 + 18 + 8 \end{bmatrix} = \begin{bmatrix} 31 \end{bmatrix}$$

$$BA=\begin{bmatrix}5\\6\\4\end{bmatrix}\begin{bmatrix}1 & 3 & 2\end{bmatrix}=\begin{bmatrix}5.1 & 5.3 & 5.2\\6.1 & 6.3 & 6.2\\4.1 & 4.3 & 4.2\end{bmatrix}=\begin{bmatrix}5 & 15 & 10\\6 & 18 & 12\\4 & 12 & 8\end{bmatrix}$$

Unlike numerical multiplication, matrix products AB and BA need not be equal.

The number 1 plays a neutral role in numerical multiplication in the sense that $1.a=a$ and $a.1=a$ for all number a. An analogous role for matrix multiplication is played by square matrices of the following types:

$$\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$ and so on.

In general, an identity matrix I is a square matrix with 1's on the main diagonal and zeros elsewhere. If it is important to stress the size of an $n\times n$ identity matrix, denoted by I_n. Identity matrix ply a neutral role with respect to matrix multiplication in the sense that :

$$AI=A \text{ and } IB=B$$

whenever the product are defined

More formally, give the definition of matrix multiplication as follow:

If $A=\begin{bmatrix}a_{ij}\end{bmatrix}$ is $m\times n$ and B is $n\times p$ the ith row of A and the jth column of B are, respectively,

$$\begin{bmatrix}a_{i1} & a_{i2} & \ldots & a_{in}\end{bmatrix} \text{ and } \begin{bmatrix}b_{1j}\\b_{2j}\\\ldots\\b_{nj}\end{bmatrix}$$

Hence, the (i,j)-entry of the product matrix AB is the dot product:

$$a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}=\sum_{k=1}^{n}a_{ik}b_{kj}$$

This is useful in verifying fact about matrix multiplication

Theorem 4

Assume that k is an arbitrary scalar and that A, B and C are matrices of sizes such that the indicated operations can be performed:

1. $IA = A, \quad BI = B$
2. $A(BC) = (AB)C$
3. $A(B+C) = AB + AC; A(B-C) = AB - AC$
4. $(B+C)A = BA + CA; (B-C)A = BA - CA$
5. $k(AB) = (kA)B = A(kB)$
6. $(AB)^T = B^T A^T$

Matrices and Linear Equations

One of the most important motivations for matrix multiplication results from its close connection with systems of linear equations.

Consider any system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

If $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$

These equations become the single matrix equation:

$$AX = B$$

This is called the matrix form of the system of equations, and B is called the constant matrix. Matrix A is called the coefficient matrix of the system of linear equations, and a column matrix X_1 is called a solution to the system if $AX_1 = B$. The matrix form is useful for formulating results about solutions of system of linear equations. Given a system $AX = B$ there is a related system:

$$AX = 0$$

called the associated homogenous system. If X_1 is a solution to $AX = B$ and if X_0 is a solution to $AX = 0$, then $X_1 + X_0$ is a solution to $AX = B$. Indeed, $AX_1 = B$ and $AX_0 = 0$, so:

$$A(X_1 + X_0) = AX_1 + AX_0 = B + 0 = B$$

This observation has a useful converse.

Theorem 6

Suppose X_1 is a particular solution to a system $AX = B$ of linear equations. Then every solution X_2 to $AX = B$ has the form :

$$X_2 = X_0 + X_1$$

for some solution X_0 of the associated homogeneous system $AX = 0$.

Proof:

Suppose that X_2 is any solution to $AX = B$ so that $AX_2 = B$. Write X_0 = X_2 - X_1, then $X_2 = X_0 + X_1$, and we compute:

$$AX_0 = A(X_2 - X_1) = AX_2 - AX_1 = B - B = 0$$

Thus X_0 is a solution to the associated homogeneous system $AX = 0$.

The important of Theorem 2 lies in the fact that sometimes a particular solution X_1 is easily to found, and so the problem of finding all solutions is reduced solving the associated homogeneous system.

Example 22 Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{aligned} x - y - z &= 2 \\ 2x - y - 3z &= 6 \\ x - 2z &= 4 \end{aligned}$$

Solution

Gaussian elimination gives $x = 4 + 2t, y = 2 + t, z = t$, where t is arbitrary. Hence the general solution is:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+2t \\ 2+t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} t$$

Thus $X_0 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ is a specific solution, and $X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} t$ gives all solutions to the associated homogeneous system (do the Gaussian elimination with all the constants zero)

Theorem 6 focuses attention on homogeneous systems. In that case there is a convenient matrix form for the solutions that will be needed later.

Example 23 Solve the homogeneous system $AX = 0$, where:

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

Solution

The reduction of the augmented matrix to reduced form is:

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the solution are $a = 2r + \frac{1}{5}t, b = r, c = \frac{3}{5}t, d = t$ by Gaussian elimination. Hence we can write the general solution X in the matrix form:

$$X = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2r + \frac{1}{5}t \\ r \\ \frac{3}{5}t \\ t \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = rX_1 + tX_2$$

Where $X_1 = \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} \frac{1}{5} & 0 & \frac{3}{5} & 1 \end{bmatrix}$ are particular solutions determined by the Gaussian Algorithm.

The solutions X_1 and X_2 in Example 23 are called the basic solutions to the homogeneous system, and a solution of the form $rX_1 + tX_2$ is called linear combination of the basic solution X_1 and X_2.

In the same way, the Gaussian algorithm produces basic solutions to every homogeneous system $AX = 0$ (there are no basic solution if there is only the trivial solution). Moreover, every solution is given by the algorithm as a linear combination of these basic solutions (as in example 23)

Exercises 2.2

1. Find a,b,c,d if :

a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 7 & 2 \\ -1 & 4 \end{bmatrix}$

2. Verify that $A^2 - A - 6I = 0$ if:

a. $A=\begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$ b. $A=\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

3. Express every solution of the system as a sum of a specific solution plus a solution of the associated homogenous system.

a. $\begin{aligned} x-y-4z&=-4 \\ x+2y+5z&=2 \\ x+y+2z&=0 \end{aligned}$ b. $\begin{aligned} 2a+b-c-d&=-1 \\ 3a+b+c-2d&=-2 \\ -a-b+2c+d&=2 \\ -2a-b+2d&=3 \end{aligned}$

4. Find the basic solutions and write the general solution as a linear combination of the basic solutions.

a. $\begin{aligned} a+2b-c+2d+e&=0 \\ a+2b+2c+e&=0 \\ 2a+4b-2c+3d+e&=0 \end{aligned}$ b. $\begin{aligned} a+b-2c+3d+2e&=0 \\ 2a-b+3c+4d+e&=0 \\ -a-2b+3c+d&=0 \\ 3a+c+7d+2e&=0 \end{aligned}$

5. Let B be an $n \times n$ matrix. Suppose $AB=0$ for some non zero $m \times n$ matrix A. Show that no $n \times n$ matrix C exists such that $BC=I$.
6. The trace of a square matrix A, denoted trA, is the sum of the elements on the main diagonal of A. Show that, if A,B are $n \times n$ matrices:

a. $tr(A+B)=trA+trB$

b. $tr(kA)=ktA$ for any number of k

c. $tr(A^T)=trA$

d. $tr(AB)=tr(BA)$

7. A square matrix is called idempotent if $P^2=P$. Show that:

a. $0, I$ are idempotents

b. If P is idempotent, so is $I - P$ and $P(I - P) = 0$

c. If P is idempotent, so is P^T

8. If P is an idempotent, so is $Q = P + AP - PAP$ for any square matrix A (of the same size as P)
9. Let A be $n \times m$ and B be $m \times n$. If $AB = I$ then BA is Idempotent.
10. Let A and B be $n \times n$ diagonal matrices (all entries off the main diagonal are zero) ,show that AB is diagonal and $AB = BA$