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Matrix Algebra

Section 2.1. Matrix Addition, Scalar Multiplication and Transposition

A rectangular array of numbers is called a matrix (the plural is matrices) and the numbers are called entries of the matrix. Matrices are usually denoted by uppercase letters: A, B, C and so on. Hence,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

are matrices. Clearly, matrices come in various shape depending on the number of rows and columns. For example, the matrix A shown has 2 rows and 3 columns. In general, a matrix with m rows and n columns is referred to as an $m \times n$ matrix or as having size $m \times n$. Thus matrices A, B, C above have sizes $2 \times 3, 2 \times 2, 3 \times 1$, respectively. A matrix of size $1 \times n$ is called a row matrix, whereas one of size $m \times 1$ is called a column matrix.

Each entry of a matrix is identified by the row and column in which it lies. The rows are numbered from the top down, and the columns are numbered from left to right. Then the (i, j) -entry of a matrix is the number lying simultaneously in row i and column j . For example:

$$\text{The } (1,2)\text{ entry of } A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \text{ is } 2$$

The $(1,2)$ -entry of $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ is -1

A special notation has been devised for the entries of a matrix. If A is an $m \times n$ matrix, and if the (i, j) -entry is denoted as a_{ij} , then A is displayed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

This is usually denoted simply as $A = [a_{ij}]$. An $n \times n$ is called a square matrix. For a square matrix, the entries $a_{11}, a_{22}, \dots, a_{nn}$ are said to lie on the main diagonal of the matrix.

Two matrices A and B are called equal (written $A = B$) if and only if :

1. They have the same size
2. Corresponding entries are equal

or can be written as $[a_{ij}] = [b_{ij}]$ means that $a_{ij} = b_{ij}$ for all i, j .

Example 11 Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, discuss the possibility

that $A = B$, $B = C$, $A = C$

Solution:

$A = B$ is impossible, because A and B are of different sizes. Similarly, $B = C$ is impossible. $A = C$ is possible provided that corresponding entries

are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ means $a = 1$, $b = -1$, $c = 0$, $d = 2$.

Matrix Addition

If A and B are matrices of the same size, their sum $A+B$ is the matrix formed by adding corresponding entries. If $A = [a_{ij}]$ and $B = [b_{ij}]$, this takes the form:

$$A+B = [a_{ij} + b_{ij}]$$

Note that addition is not defined for matrices of different sizes.

Example 12 If $A = \begin{bmatrix} 1 & -1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix}$, compute $A+B$!

Solution

$$A+B = \begin{bmatrix} 1+1 & -1+2 & -1-1 \\ 3+0 & -2+5 & 4+6 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 3 & 10 \end{bmatrix}$$

Example 13 Find a, b, c if $\begin{bmatrix} 1 & b & c \end{bmatrix} + \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}$

Solution

Add the matrices on the left side to obtain:

$$\begin{bmatrix} 1+c & b+a & c+b \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

Because the corresponding entries must be equal, this gives three equations:

$$a+c=3, b+a=2, c+b=-1. \text{ Solving these yields } a=3, b=-1, c=0.$$

The properties of Matrix Addition

If A, B, C are any matrices of the same size, then:

1. $A+B=B+A$ (commutative law)
2. $A+(B+C)=(A+B)+C$ (Associative law)

The $m \times n$ matrix in which every entry is zero is called the zero matrix and is denoted as O , hence,

$$3. \quad 0 + X = X$$

The negative of an $m \times n$ matrix A (written as $-A$) is defined to be $m \times n$ matrix obtained by multiply each entry of A by -1 . If $A = [a_{ij}]$, this becomes $-A = [-a_{ij}]$, hence,

$$4. \quad A + (-A) = 0$$

for all matrices $A = [a_{ij}]$ where 0 is the zero matrix of the same size as A .

A closely related notion is that of subtracting matrices. If A, B are two $m \times n$ matrices, their difference $A - B$ is defined by:

$$A - B = A + (-B), \text{ i.e. : } A - B = [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}]$$

Example 14 $A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Compute $-A, A - B, A + B - C$

Solution

$$-A = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 - 3 & 1 + 2 \\ 0 - 2 & 1 - 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ -2 & 0 \end{bmatrix}$$

$$A + B - C = \begin{bmatrix} -2 + 3 - 1 & 1 - 2 - 1 \\ 0 + 2 - 2 & 1 + 1 - 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

Example 15 Solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, where X is a matrix.

Solution 1

X must be a 2×2 matrix. If $X = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$, the equation reads:

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 3+x & 2+y \\ -1+u & 1+v \end{bmatrix}$$

The rule of matrix equality gives $1=3+x$, $0=2+y$, $-1=-1+u$, $2=1+v$.

$$\text{Thus } X = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}.$$

Solution 2

We solve a numerical equation $a+x=b$ by subtracting the number a from both sides to obtain $x=b-a$. This also works for matrices. To solve

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \text{ simply subtract the matrix } \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \text{ from both sides}$$

to get:

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

Scalar Multiplication

In Gaussian Elimination, multiplying a row of matrix by a number k means multiplying every entry of that row by k . More generally, if A is any matrix and k is any number, the scalar multiple kA is the matrix obtained from A by multiplying each entry of A by k . If $A = [a_{ij}]$, this is:

$$kA = [ka_{ij}]$$

We have been using real numbers as scalars, but we could equally well have been using complex numbers.

Example 16 If $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, compute $5A$, $3B-2A$

Solution

$$5A = 5 \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 \times 3 & 5 \times 2 \\ 5 \times (-1) & 5 \times 1 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ -5 & 5 \end{bmatrix}$$

$$3B - 2A = 3 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - 2 \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 - 6 & 0 - 4 \\ -3 + 2 & 2 - 2 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -1 & 0 \end{bmatrix}$$

If A is any matrix, note that kA is the same size as A for all scalars k . We also have: $0A = 0$ and $k0 = 0$

Because the zero matrix has every entry zero. In other words, $kA = 0$ if either $k = 0$ or $A = 0$.

The properties of scalar multiplication

Let A, B denote arbitrary $m \times n$ matrices, where m, n are fixed, let k, l denote arbitrary real numbers. Then :

1. $k(A + B) = kA + kB$
2. $(k + l)A = kA + lA$
3. $(kl)A = k(lA)$
4. $1.A = A$

Example 17 Simplify $2(A + 3C) - 3(2C - B) - 3[2(2A + B - 4C) - 4(A - 2C)]$ where A, B, C are all matrices of the same size.

Solution

$$\begin{aligned} & 2(A + 3C) - 3(2C - B) - 3[2(2A + B - 4C) - 4(A - 2C)] \\ &= 2A + 6C - 6C + 3B - 12A - 6B + 24C + 12A - 24C \\ &= 2A - 3B \end{aligned}$$

Transpose

Many result about a matrix A involve the rows of A , and the corresponding result for columns is derived in an analogous way, essentially by replacing

the word row by the word column throughout. The following definition is made with such application in mind.

If A is an $m \times n$ matrix, the transpose of A , written as A^T , is the $n \times m$ matrix whose rows are just the columns of A in the same order.

In other words, the first row of A^T is the first column of A , the second row of A^T is the second column of A , and so on.

Example 18 Write down the transpose of each of the following matrices:

$$A = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}, B^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C^T = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}, D^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$$

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is a matrix, write $A^T = \begin{bmatrix} b_{ij} \end{bmatrix}$. Then b_{ij} is the j th element of the i th row of A^T and so is the j th element of the i th column of A . This means $b_{ij} = a_{ji}$ so the definition of A^T can be stated as follows:

$$\text{If } A = \begin{bmatrix} a_{ij} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{ji} \end{bmatrix}$$

The Properties of transposition:

Theorem 3 Let A, B denote matrices of the same size, and let k denote a scalar.

1. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.
2. $(A^T)^T = A$
3. $(kA)^T = kA^T$

$$4. (A+B)^T = A^T + B^T$$

The matrix D in Example 8 has the property that $D = D^T$. Such matrices are important.

A matrix A is called symmetric if $A = A^T$.

A symmetric matrix A is necessarily square. The name comes from the fact that these matrices exhibit a symmetry about the main diagonal. That is, entries that are directly across the main diagonal from each other are equal.

Example 19 If A, B are symmetric $n \times n$, show that $A + B$ is symmetric

Solution

We have $A = A^T$ and $B = B^T$, so by Theorem 1, we obtain
 $(A+B)^T = A^T + B^T$

Exercises 2.1:

1. Find a, b, c, d if:

$$a. \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c-3d & -d \\ 2a+d & a+b \end{bmatrix}$$

$$b. 3 \begin{bmatrix} a \\ b \end{bmatrix} + 2 \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$2. \text{ Let } A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Compute the following (if possible)

$$a. 3A - 2B$$

$$b. B^T - 2D + 3E^T$$

$$c. (A+C)^T$$

$$d. 2A^T - 4D$$

$$e. 2D^T + 3E$$

3. If X, Y, A and B are matrices of the same size, solve the following equations to obtain X, Y in terms of A, B :

$$a. 5X + 3Y = A$$

$$b. 4X + 3Y = A$$

$$2X + Y = B$$

$$5X + 4Y = B$$

4. A square matrix B is called skew-symmetric if $B^T = -B$. Let A be any square matrix.
 - a. Show that $A - A^T$ is skew-symmetric
 - b. Find a symmetric matrix S and a skew-symmetric matrix W such that $A = S + W$
 - c. If W is skew-symmetric show that the entries on the main diagonal are zero
5. A square matrix is called a diagonal matrix if all entries off the main diagonal are zero. If A, B are diagonal matrices show that $A + B, A - B, kA$ are diagonal matrices.
6. Let A be any square matrix. If $A = pB^T$ and $B = qA^T$ for some matrix B and numbers p, q , show that either $A = 0$ or $p \cdot q = 1$