Elementary Linear Algebra

References:

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Chapter Contents

1.1 Introduction to Systems of Equations

Linear Equations

- Any straight line in xy-plane can be represented algebraically by an equation of the form: $a_1x + a_2y = b$
- General form: define a linear equation in the n variables $x_1, x_2, \ldots, x_n$:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  - Where $a_1, a_2, \ldots, a_n,$ and b are real constants.
  - The variables in a linear equation are sometimes called unknowns.

Example 1
Linear Equations

- The equations $x + 3y = 7, y = \frac{1}{2}x + 3z + 1,$ and $x_1 - 2x_2 - 3x_3 + x_4 = 7$ are linear.
- Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.
- The equations $x + 3\sqrt{y} = 5, 3x + 2y - z + xz = 4,$ and $y = \sin x$ are not linear.
  - A solution of a linear equation is a sequence of n numbers $s_1, s_2, \ldots, s_n$ such that the equation is satisfied. The set of all solutions of the equation is called its solution set or general solution of the equation

Example 2
Finding a Solution Set (1/2)

- Find the solution of $\text{(a)}\ 4x - 2y = 1$

- Solution(a)

  we can assign an arbitrary value to x and solve for y , or choose an arbitrary value for y and solve for x .If we follow the first approach and assign x an arbitrary value ,we obtain $x = t_1,\ \ y = 2t_1 - \frac{1}{2}$ or $x = \frac{1}{2}t_2 + \frac{1}{4},\ \ y = t_2$

  - arbitrary numbers $t_{1,}\ t_2$ are called parameter.
  - for example

    $t_1 = 3$ yields the solution $x = 3, y = \frac{11}{2}$ as $t_2 = \frac{11}{2}$

Example 2
Finding a Solution Set (2/2)

- Find the solution of (b) $x_1 - 4x_2 + 7x_3 = 5.$

- Solution(b)

  we can assign arbitrary values to any two variables and solve for the third variable.

  - for example

    $x_1 = 5 + 4s - 7t, \qquad x_2 = s, \qquad x_3 = t$

  - where s, t are arbitrary values

Linear Systems (1/2)

- A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations or a linear system .

- A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a solution of the system.

- A system has no solution is said to be inconsistent ; if there is at least one solution of the system, it is called consistent.

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m
\end{array}
$$

$\uparrow$ An arbitrary system of m linear equations in n unknowns

Linear Systems (2/2)

- Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.

- A general system of two linear equations:
(Figure1.1.1) $a_1x + b_1y = c_1$ (a_1, b_1 not both zero)
$a_2x + b_2y = c_2$ (a_2, b_2 not both zero)
  - Two lines may be parallel -> no solution
  - Two lines may intersect at only one point -> one solution
  - Two lines may coincide -> infinitely many solution

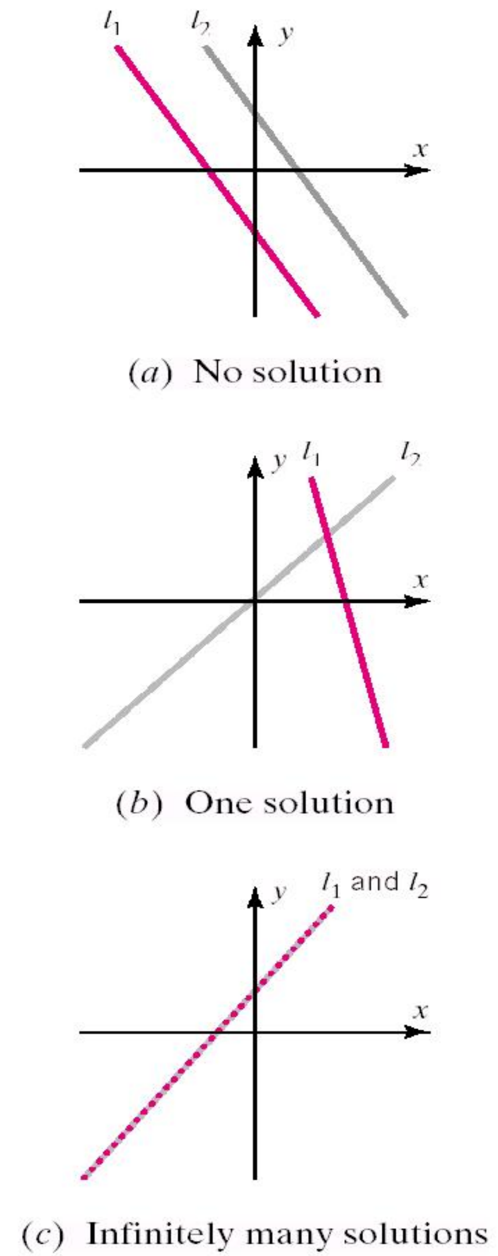


Figure 1.1.1

Augmented Matrices

- The location of the +'s, the x's, and the ='s can be abbreviated by writing only the rectangular array of numbers.
- This is called the augmented matrix for the system.
- Note: must be written in the same order in each equation as the unknowns and the constants must be on the right.

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}
$$

$$
\begin{array}{ll}
\text{1th column} & \\
\downarrow & \\
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m
\end{array}\right] & \leftarrow \text{1th row}
\end{array}
$$

Elementary Row Operations

- The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but which is easier to solve.

- Since the rows of an augmented matrix correspond to the equations in the associated system. new systems is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically. These are called elementary row operations.

  1. Multiply an equation through by an nonzero constant.
  2. Interchange two equation.
  3. Add a multiple of one equation to another.

Example 3
Using Elementary row Operations(1/4)

$$\begin{aligned} x + \quad y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \xrightarrow[\text{to the second}]{\substack{\text{add -2 times} \\ \text{the first equation}}} \begin{aligned} x + \quad y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \xrightarrow[\text{to the third}]{\substack{\text{add -3 times} \\ \text{the first equation}}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow[\text{to the second}]{\substack{\text{add -2 times} \\ \text{the first row}}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow[\text{to the third}]{\substack{\text{add -3 times} \\ \text{the first row}}}$$

Example 3
Using Elementary row Operations(2/4)

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \xrightarrow{\text{multiply the second equation by } \frac{1}{2}} \begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= 0 \end{aligned} \xrightarrow{\text{add -3 times the second equation to the third}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \xrightarrow{\text{multily the second row by } \frac{1}{2}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\tfrac{7}{2} & -\tfrac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix} \xrightarrow{\text{add -3 times the second row to the third}}$$

Example 3
Using Elementary row Operations(3/4)

$$\begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned} \xrightarrow{\text{Multiply the third equation by -2}} \begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \xrightarrow{\text{Add -1 times the second equation to the first}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \xrightarrow{\text{Multily the third row by -2}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{Add -1 times the second row to the first}}$$

Example 3
Using Elementary row Operations(4/4)

$$\begin{aligned} x \qquad + \tfrac{11}{2} z &= \tfrac{35}{2} \\ y - \tfrac{7}{2} z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \xrightarrow[\text{to the second}]{\substack{\text{Add } -\frac{11}{2} \text{ times} \\ \text{the third equation} \\ \text{to the first and } \frac{7}{2} \text{ times} \\ \text{the third equation}}} \begin{aligned} x \qquad &= 1 \\ y \qquad &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[\text{to the second}]{\substack{\text{Add } -\frac{11}{2} \text{ times} \\ \text{the third row} \\ \text{to the first and } \frac{7}{2} \\ \text{times the third row}}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- The solution x=1,y=2,z=3 is now evident.

1.2 Gaussian Elimination

Echelon Forms

- This matrix which have following properties is in reduced row-echelon form

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leader 1.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leader 1 in the lower row occurs farther to the right than the leader 1 in the higher row.
4. Each column that contains a leader 1 has zeros everywhere else.

- A matrix that has the first three properties is said to be in row-echelon form (Example 1, 2).
- A matrix in reduced row-echelon form is of necessity in row-echelon form, but not conversely.

Example 1
Row-Echelon & Reduced Row-Echelon form

- reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- row-echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2
More on Row-Echelon and Reduced Row-Echelon form

- All matrices of the following types are in row-echelon form (any real numbers substituted for the *'s.) :

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- All matrices of the following types are in reduced row-echelon form (any real numbers substituted for the *'s.) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Example 3
Solutions of Four Linear Systems (a)

Suppose that the augmented matrix for a system of linear equations have been reduced by row operations to the given reduced row-echelon form. Solve the system.

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Solution (a)

the corresponding system of equations is : $\longrightarrow$

$$\begin{aligned} x \qquad\qquad &= 5 \\ y \qquad &= -2 \\ z &= 4 \end{aligned}$$

Example 3
Solutions of Four Linear Systems (b1)

$$(b)\ \begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

Solution (b)

1. The corresponding system of equations is :

$$\begin{aligned} x_1 \qquad\qquad + 4x_4 &= -1 \\ x_2 \qquad + 2x_4 &= 6 \\ x_3 + 3x_4 &= 2 \end{aligned}$$

free variables

leading variables

Example 3
Solutions of Four Linear Systems (b2)

$$
\begin{aligned}
x_1 &= -1 - 4x_4 \\
x_2 &= 6 - 2x_4 \\
x_3 &= 2 - 3x_4
\end{aligned}
$$

2. We see that the free variable can be assigned an arbitrary value, say t, which then determines values of the leading variables.

3. There are infinitely many solutions, and the general solution is given by the formulas

$$
\begin{aligned}
x_1 &= -1 - 4t, \\
x_2 &= 6 - 2t, \\
x_3 &= 2 - 3t, \\
x_4 &= t
\end{aligned}
$$

Example 3
Solutions of Four Linear Systems (c1)

$$\text{(c)} \begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution (c)

1. The 4th row of zeros leads to the equation places no restrictions on the solutions (why?). Thus, we can omit this equation. $\longrightarrow$

$$\begin{aligned} x_1 + 6x_2 \qquad\qquad + 4x_5 &= -2 \\ x_3 \qquad + 3x_5 &= 1 \\ x_4 + 5x_5 &= 2 \end{aligned}$$

Example 3
Solutions of Four Linear Systems (c2)

Solution (c)

2. Solving for the leading variables in terms of the free variables: $\rightarrow$

$$x_1 = -2 - 6x_2 - 4x_5$$
$$x_3 = 1 - 3x_5$$
$$x_4 = 2 - 5x_5$$

3. The free variable can be assigned an arbitrary value,there are infinitely many solutions, and the general solution is given by the formulas. $\longrightarrow$

$$x_1 = -2 - 6s - 4t\,,$$
$$x_2 = s$$
$$x_3 = 1 - 3t$$
$$x_4 = 2 - 5t,$$
$$x_4 = t$$

Example 3
Solutions of Four Linear Systems (d)

$$\text{(d)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution (d):

the last equation in the corresponding system of equation is

$$0x_1 + 0x_2 + 0x_3 = 1$$

Since this equation cannot be satisfied, there is no solution to the system.

Elimination Methods (1/7)

- We shall give a step-by-step elimination procedure that can be used to reduce any matrix to reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Elimination Methods (2/7)

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1th and 2th rows in the preceding matrix were interchanged.

Elimination Methods (3/7)

- Step3. If the entry that is now at the top of the column found in Step1 is a, multiply the first row by 1/a in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1st row of the preceding matrix was multiplied by 1/2.

- Step4. Add suitable multiples of the top row to the rows below so that all entires below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-2 times the 1st row of the preceding matrix was added to the 3rd row.

Elimination Methods (4/7)

- Step5. Now cover the top row in the matrix and begin again with Step1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form.

$$\begin{bmatrix} \boxed{1 & 2 & -5 & 3 & 6 & 14} \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & -5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The 1st row in the submatrix was multiplied by -1/2 to introduce a leading 1.

Elimination Methods (5/7)

- Step5 (cont.)

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

-5 times the 1st row of the submatrix was added to the 2nd row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again Step1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submetrix was multiplied by 2 to introduce a leading 1.

- The entire matrix is now in row-echelon form.

Elimination Methods (6/7)

- Step6. Beginning with las nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

7/2 times the 3rd row of the preceding matrix was added to the 2nd row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the 3rd row was added to the 1st row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the 2nd row was added to the 1st row.

- The last matrix is in reduced row-echelon form.

Elimination Methods (7/7)

- Step1~Step5: the above procedure produces a row-echelon form and is called Gaussian elimination.
- Step1~Step6: the above procedure produces a reduced row-echelon form and is called Gaussian-Jordan elimination.
- Every matrix has a unique reduced row-echelon form but a row-echelon form of a given matrix is not unique.

Example 4
Gauss-Jordan Elimination(1/4)

- Solve by Gauss-Jordan Elimination

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2\mathrm{x}_5 \qquad\quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 5 \\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 - 18x_6 &= 6
\end{aligned}$$

- Solution:

The augmented matrix for the system is

$$\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{bmatrix}$$

Example 4
Gauss-Jordan Elimination(2/4)

- Adding -2 times the 1st row to the 2nd and 4th rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

- Multiplying the 2nd row by -1 and then adding -5 times the new 2nd row to the 3rd row and -4 times the new 2nd row to the 4th row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Example 4
Gauss-Jordan Elimination(3/4)

- Interchanging the 3rd and 4th rows and then multiplying the 3rd row of the resulting matrix by 1/6 gives the row-echelon form.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Adding -3 times the 3rd row to the 2nd row and then adding 2 times the 2nd row of the resulting matrix to the 1st row yields the reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4
Gauss-Jordan Elimination(4/4)

- The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2\mathrm{x}_5 \quad &= 0 \\ x_3 + 2x_4 \quad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- Solution

The augmented matrix for the system is

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2\mathrm{x}_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- We assign the free variables, and the general solution is given by the formulas:

$$x_1 = -3r - 4s - 2t,\ \ x_2 = r,\ \ x_3 = -2s,\ \ x_4 = s,\ \ x_5 = t,\ \ x_6 = \tfrac{1}{3}$$

Back-Substitution

- It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form without continuing all the way to the reduced row-echelon form.
- When this is done, the corresponding system of equations can be solved by solved by a technique called back-substitution.
- Example 5

Example 5
ex4 solved by Back-substitution(1/2)

- From the computations in Example 4, a row-echelon form from the augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2\mathrm{x}_5 \quad &= 0 \\ x_3 + 2x_4 \quad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- Step1. Solve the equations for the leading variables.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2\mathrm{x}_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Example5
ex4 solved by Back-substitution(2/2)

- Step2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.
  - Substituting x6=1/3 into the 2nd equation

$$x_1 = -3x_2 + 2x_3 - 2\mathrm{x}_5$$
$$x_3 = -2x_4$$
$$x_6 = \tfrac{1}{3}$$

  - Substituting x3=-2 x4 into the 1st equation

$$x_1 = -3x_2 + 2x_3 - 2\mathrm{x}_5$$
$$x_3 = -2x_4$$
$$x_6 = \tfrac{1}{3}$$

- Step3. Assign free variables, the general solution is given by the formulas.

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ \ x_6 = \tfrac{1}{3}$$

Example 6
Gaussian elimination(1/2)

- Solve $$\begin{aligned} x+\ \ y+2z&=9\\ 2x+4y-3z&=1\\ 3x+6y-5z&=0 \end{aligned}$$ by Gaussian elimination and back-substitution. (ex3 of Section1.1)
- Solution
  - We convert the augmented matrix $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$
  - to the ow-echelon form $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$
  - The system corresponding to this matrix is

$$x+y+2z=9,\ \ y-\tfrac{7}{2}z=-\tfrac{17}{2},\ \ z=3$$

Example 6
Gaussian elimination(2/2)

- Solution
  - Solving for the leading variables

$$x = 9 - y - 2z,$$
$$y = -\tfrac{17}{2} + \tfrac{7}{2}z,$$
$$z = 3$$

  - Substituting the bottom equation into those above

$$x = 3 - y,$$
$$y = 2,$$
$$z = 3$$

  - Substituting the 2nd equation into the top

$$x = 1,\ y = 2,\ z = 3$$

Homogeneous Linear Systems(1/2)

- A system of linear equations is said to be homogeneous if the constant terms are all zero; that is , the system has the form :

$$\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0
\end{array}$$

- Every homogeneous system of linear equation is consistent, since all such system have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the trivial solution; if there are another solutions, they are called nontrivial solutions.
- There are only two possibilities for its solutions:
  - The system has only the trivial solution.
  - The system has infinitely many solutions in addition to the trivial solution.

Homogeneous Linear Systems(2/2)

- In a special case of a homogeneous linear system of two linear equations in two unknowns: (fig1.2.1)

$$a_1x + b_1y = 0 \; (a_1, b_1 \text{ not both zero})$$
$$a_2x + b_2y = 0 \; (a_2, b_2 \text{ not both zero})$$

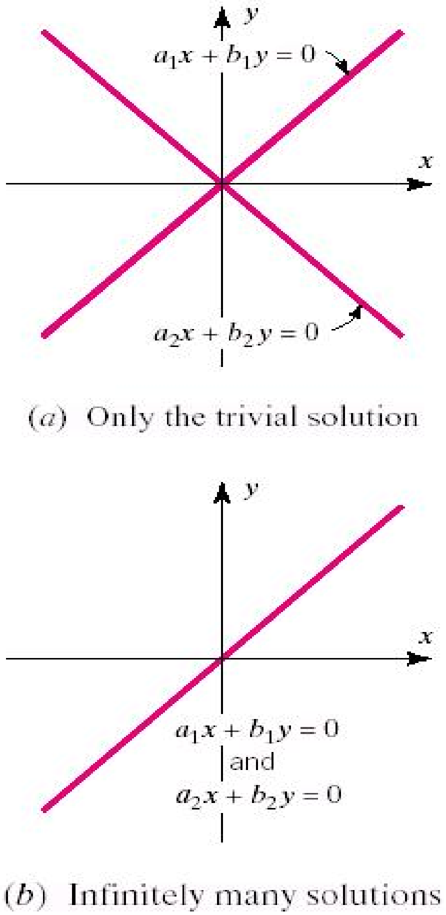


Figure 1.2.1

Example 7
Gauss-Jordan Elimination(1/3)

- Solve the following homogeneous system of linear equations by using Gauss-Jordan elimination.

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \qquad\quad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \qquad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

- Solution
  - The augmented matrix

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

  - Reducing this matrix to reduced row-echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 7
Gauss-Jordan Elimination(2/3)

Solution (cont)

- The corresponding system of equation

$$\begin{aligned} x_1 + x_2 \qquad\qquad + x_5 &= 0 \\ x_3 \quad + x_5 &= 0 \\ x_4 \qquad &= 0 \end{aligned}$$

- Solving for the leading variables is

$$\begin{aligned} x_1 &= - x_2 - x_5 \\ x_3 &= -x_5 \\ x_4 &= 0 \end{aligned}$$

- Thus the general solution is

$$x_1 = - s - t\ ,\ \ x_2 = s,\ \ x_3 = -t,\ \ x_4 = 0,\ \ x_5 = t$$

- Note: the trivial solution is obtained when s=t=0.

Example7
Gauss-Jordan Elimination(3/3)

- Two important points:
  - Non of the three row operations alters the final column of zeros, so the system of equations corresponding to the reduced row-echelon form of the augmented matrix must also be a homogeneous system.
  - If the given homogeneous system has m equations in n unknowns with m<n, and there are r nonzero rows in reduced row-echelon form of the augmented matrix, we will have r<n. It will have the form:

$$
\begin{array}{lllll}
\Lambda \;\; x_{k1} \qquad\qquad + \sum () = 0 & & & x_{k1} = -\sum () & \\
\qquad \Lambda \;\; x_{k2} \qquad + \sum () = 0 & & & x_{k2} = -\sum () & \\
\qquad\qquad\qquad O \qquad M & & & M & \\
\qquad\qquad\qquad\qquad x_r + \sum () = 0 & (1) & & x_r = -\sum () & (2)
\end{array}
$$

Theorem 1.2.1

A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

- Note: theorem 1.2.1 applies only to homogeneous system
- Example 7 (3/3)

Computer Solution of Linear System

- Most computer algorithms for solving large linear systems are based on Gaussian elimination or Gauss-Jordan elimination.
- Issues
  - Reducing roundoff errors
  - Minimizing the use of computer memory space
  - Solving the system with maximum speed

1.3 Matrices and Matrix Operations

Definition

A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

Example 1
Examples of matrices

- Some examples of matrices

entries

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} \lambda & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}$$

row matrix or row vector

column matrix or column vector

- Size

3 x 2, 1 x 4, 3 x 3, 2 x 1, 1 x 1

columns

rows

Matrices Notation and Terminology(1/2)

- A general m x n matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} or $(A)_{ij}$. If a_{ij} is real number, it is common to be referred as scalars.

Matrices Notation and Terminology(2/2)

- The preceding matrix can be written as

$$\left[a_{ij}\right]_{m\times n} \quad \text{or} \quad \left[a_{ij}\right]$$

- A matrix A with n rows and n columns is called a square matrix of order n, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ are said to be on the main diagonal of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Definition

Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size,

then $\mathrm{A} = \mathrm{B}$ if and only if $a_{ij} = b_{ij}$ for all i and j.

Example 2
Equality of Matrices

- Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

  - If x=5, then A=B.
  - For all other values of x, the matrices A and B are not equal.
  - There is no value of x for which A=C since A and C have different sizes.

Operations on Matrices

- If A and B are matrices of the same size, then the sum A+B is the matrix obtained by adding the entries of B to the corresponding entries of A.
- Vice versa, the difference A-B is the matrix obtained by subtracting the entries of B from the corresponding entries of A.
- Note: Matrices of different sizes cannot be added or subtracted.

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$$

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Example 3
Addition and Subtraction

- Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

  - Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix}, \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

  - The expressions A+C, B+C, A-C, and B-C are undefined.

Definition

If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be the scalar multiple of A.

In matrix notation, if $A = \lfloor a_{ij} \rfloor$, then

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

Example 4
Scalar Multiples (1/2)

- For the matrices

$$A=\begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & 2 & 7\\ -1 & 3 & -5\end{bmatrix},\quad C=\begin{bmatrix}9 & -6 & 3\\ 3 & 0 & 12\end{bmatrix}$$

  - We have

$$2A=\begin{bmatrix}4 & 6 & 8\\ 2 & 6 & 2\end{bmatrix},\quad (-1)B=\begin{bmatrix}0 & -2 & -7\\ 1 & -3 & 5\end{bmatrix},\quad \tfrac{1}{3}C=\begin{bmatrix}3 & -2 & 1\\ 1 & 0 & 4\end{bmatrix}$$

  - It common practice to denote (-1)B by –B.

Example 4
Scalar Multiples (2/2)

If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

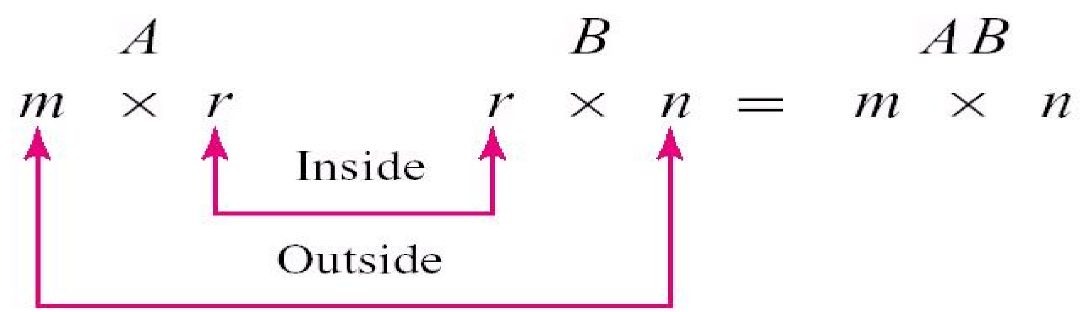
is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with ***coefficients*** $c_1, c_2, \ldots, c_n$. For example, if A, B, and C are the matrices in Example 4, then

$$\begin{aligned} 2A - B + \tfrac{1}{3}C &= 2A + (-1)B + \tfrac{1}{3}C \\ &= \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 2 \\ 4 & 3 & 11 \end{bmatrix} \end{aligned}$$

is the linear combination of A, B, and C with scalar coefficients 2, -1, and $\frac{1}{3}$.

Definition

- If A is an m×r matrix and B is an r×n matrix, then the product AB is the m×n matrix whose entries are determined as follows.
- To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B .Multiply the corresponding entries from the row and column together and then add up the resulting products.



Example 5
Multiplying Matrices (1/2)

- Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

- Solution
  - Since A is a 2 ×3 matrix and B is a 3 ×4 matrix, the product AB is a 2 ×4 matrix. And:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \tag{5}$$

Example 5
Multiplying Matrices (2/2)

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining products are

$$\begin{aligned} (1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\ (1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\ (1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\ (2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\ (2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\ (2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12 \end{aligned}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Examples 6
Determining Whether a Product Is Defined

- Suppose that A ,B ,and C are matrices with the following sizes:

A	B	C
3×4	4×7	7×3

- Solution:
  - Then by (3), AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix; and CA is defined and is a 7×4 matrix. The products AC ,CB ,and BA are all undefined.

Partitioned Matrices

- A matrix can be subdivided or partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns.

- For example, below are three possible partitions of a general 3×4 matrix A.
  - The first is a partition of A into four submatrices A_{11}, A_{12}, A_{21}, and A_{22}.
  - The second is a partition of A into its row matrices r_1, r_2, and r_3.
  - The third is a partition of A into its column matrices c_1, c_2, c_3, and c_4.

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

Matrix Multiplication by columns and by Rows

- Sometimes it may b desirable to find a particular row or column of a matrix product AB without computing the entire product.

$$j\text{th column matrix of } AB = A[j\text{th column matrix of } B] \tag{6}$$

$$i\text{th row matrix of } AB = [i\text{th row matrix of } A]B \tag{7}$$

- If a 1 ,a 2 ,…,a m denote the row matrices of A and b 1 ,b 2, …,b n denote the column matrices of B ,then it follows from Formulas (6)and (7)that

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$

(AB computed column by column)

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

(AB computed row by row)

Example 7
Example5 Revisited

- This is the special case of a more general procedure for multiplying partitioned matrices.
- If A and B are the matrices in Example 5,then from (6)the second column matrix of AB can be obtained by the computation

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{Second column} \\ \text{of } B}}{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{Second column} \\ \text{of } AB}}{\begin{bmatrix} 27 \\ -4 \end{bmatrix}}$$

- From (7) the first row matrix of AB can be obtained by the computation

$$\underset{\text{First row of } A}{\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \underset{\text{First row of } AB}{\begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}}$$

Matrix Products as Linear Combinations (1/2)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{10}$$

Matrix Products as Linear Combinations (2/2)

- In words, (10)tells us that the product $A\ x$ of a matrix A with a column matrix x is a linear combination of the column matrices of A with the coefficients coming from the matrix x .
- In the exercises w ask the reader to show that the product $y\ A$ of a $1\times m$ matrix y with an $m\times n$ matrix A is a linear combination of the row matrices of A with scalar coefficients coming from y .

Example 8
Linear Combination

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the linear combination of column matrices

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

The matrix product

$$\begin{bmatrix} 1 & -9 & -3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

can be written as the linear combination of row matrices

$$1\begin{bmatrix} -1 & 3 & 2 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

Example 9
Columns of a Product AB as Linear Combinations

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of AB can be expressed as linear combinations of the column matrices of A as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Matrix form of a Linear System(1/2)

- Consider any system of m linear equations in n unknowns.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

- Since two matrices are equal if and only if their corresponding entries are equal.

$$
\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

- The m×1 matrix on the left side of this equation can be written as a product to give:

$$
\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Matrix form of a Linear System(1/2)

- If w designate these matrices by A ,x ,and b ,respectively, the original system of m equations in n unknowns has been replaced by the single matrix equation $A\mathbf{x} = \mathbf{b}$
- The matrix A in this equation is called the coefficient matrix of the system. The augmented matrix for the system is obtained by adjoining b to A as the last column; thus the augmented matrix is

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition

- If A is any m×n matrix, then the transpose of A ,denoted by A^T,is defined to be the n×m matrix that results from interchanging the rows and columns of A ; that is, the first column of A^T is the first row of A ,the second column of A^T is the second row of A ,and so forth.

Example 10
Some Transposes (1/2)

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

Example 10
Some Transposes (2/2)

- Observe that $(A^T)_{ij} = (A)_{ji}$

- In the special case where A is a square matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Definition

- If A is a square matrix, then the trace of A ,denoted by tr(A), is defined to be the sum of the entries on the main diagonal of A .The trace of A is undefined if A is not a square matrix.

Example 11
Trace of Matrix

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Determinants

1. Determinants by Cofactor Expansion

2. Evaluating Determinants by Row Reduction

3. Properties of Determinants; Cramer's Rule

1 Determinants by Cofactor Expansion

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{5}$$

[cofactor expansion along the *j*th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{6}$$

[cofactor expansion along the *i*th row]

▶ EXAMPLE 3 Cofactor Expansion Along the First Row

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

▶ EXAMPLE 4 Cofactor Expansion Along the First Column

Let A be the matrix in Example 3, and evaluate $\det(A)$ by cofactor expansion along the first column of A.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

This agrees with the result obtained in Example 3.

A technique for determinants of 2x2 and 3x3 matrices <u>only</u>

EXAMPLE 7 **A Technique for Evaluating 2 × 2 and 3 × 3 Determinants**

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix}\begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

2. Row Reduction and Determinants

THEOREM 2.2.3 *Let A be an $n \times n$ matrix.*

(*a*) *If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then* $\det(B) = k\det(A)$.

(*b*) *If B is the matrix that results when two rows or two columns of A are interchanged, then* $\det(B) = -\det(A)$.

(*c*) *If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then* $\det(B) = \det(A)$.

Table 1

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of A is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of A are interchanged.
$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	A multiple of the second row of A is added to the first row.

3. Cramer's Rule

THEOREM 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ *is a system of* n *linear equations in* n *unknowns such that* $\det(A) \neq 0$, *then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j *is the matrix obtained by replacing the entries in the* j*th column of* A *by the entries in the matrix*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer's Rule

EXAMPLE 8 Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

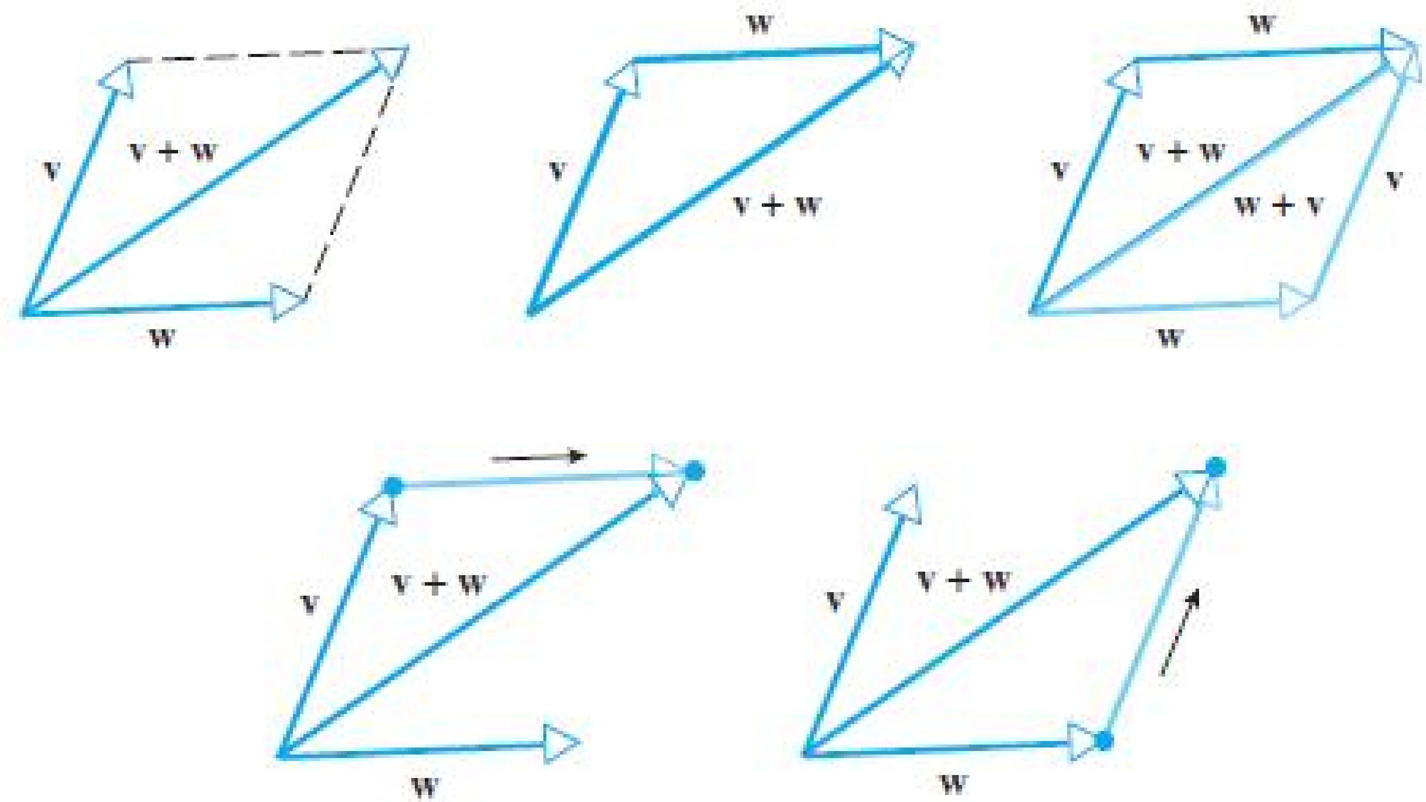
$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Euclidean Vector Spaces

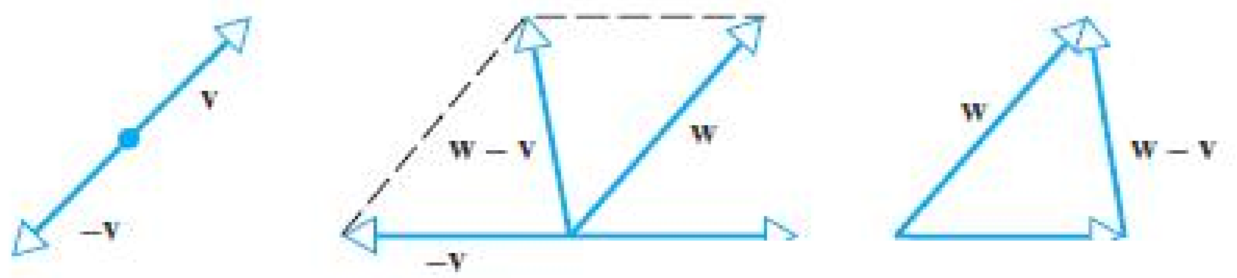
- 1 Vectors in 2-Space, 3-Space,
  and n-Space
- 2 Norm, Dot Product, and Distance in R^n
- 3 Orthogonality
- 4 The Geometry of Linear Systems
- 5 Cross Product

1. Vectors

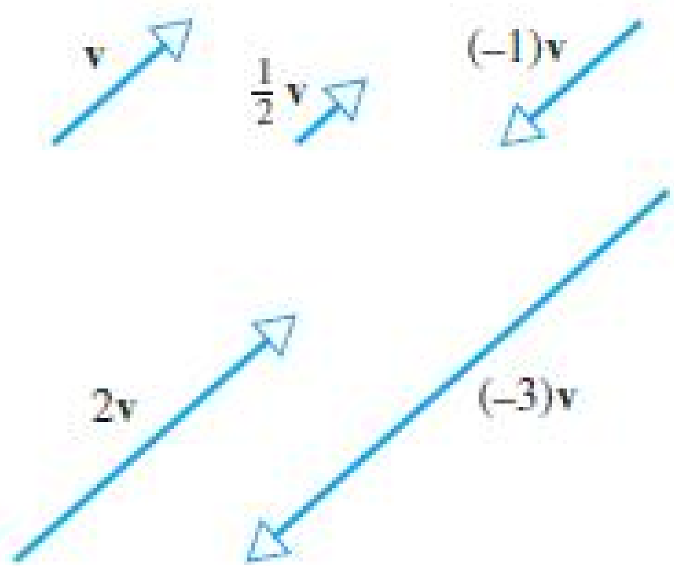
Addition of vectors by the parallelogram or triangle rules



Subtraction:



Scalar Multiplication:



Properties of Vectors

THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*f*) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(*g*) $k(m\mathbf{u}) = (km)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

Section 3.2 Norm, Dot Product, and Distance in R^n

N

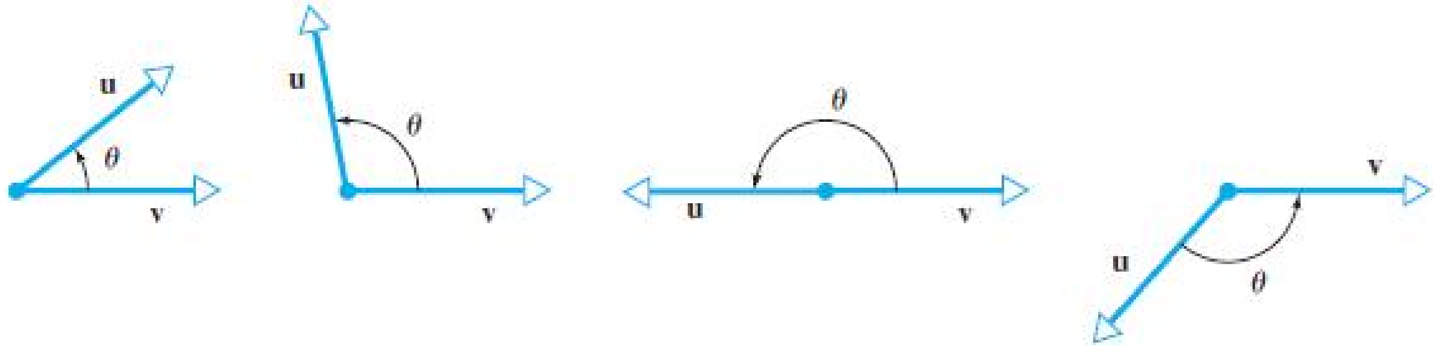
DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

Unit Vectors:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

The Dot Product



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$.

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{13}$$

The Dot Product

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{17}$$

Properties of the Dot Product

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Cauchy-Schwarz Inequality

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

Dot Products and Matrices

Table 1

Form	Dot Product	Example	
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{v}\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{u}\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

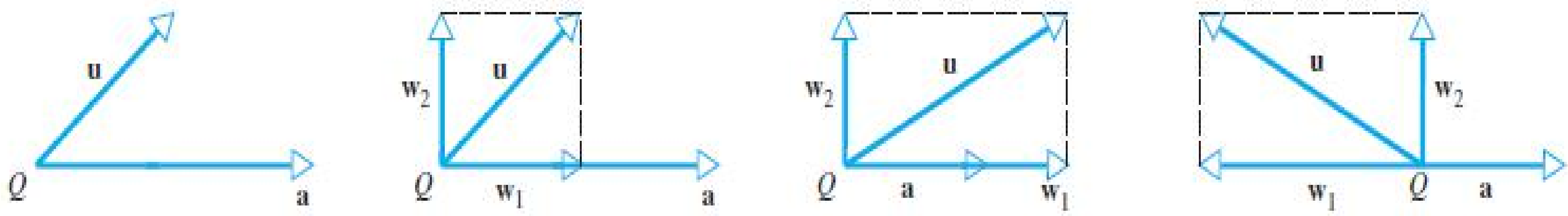
3 Orthogonality

DEFINITION 1 Two nonzero vectors **u** and **v** in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n. A nonempty set of vectors in R^n is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Orthogonal Projections

THEOREM 3.3.2 Projection Theorem

If $\mathbf{u}$ *and* $\mathbf{a}$ *are vectors in* R^n, *and if* $\mathbf{a} \neq 0$, *then* $\mathbf{u}$ *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* $\mathbf{a}$ *and* $\mathbf{w}_2$ *is orthogonal to* $\mathbf{a}$.



Point-line and point-plane Distance formulas

THEOREM 3.3.4

(a) *In* R^2 *the distance* D *between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(b) *In* R^3 *the distance* D *between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$

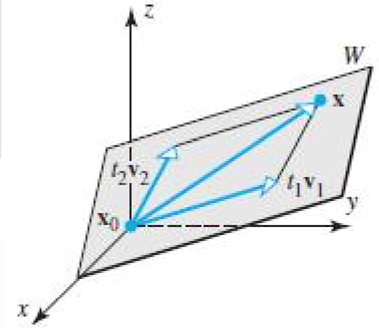
4. The Geometry of Linear Systems

THEOREM 3.4.1 *Let L be the line in R^2 or R^3 that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{1}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v} \tag{2}$$

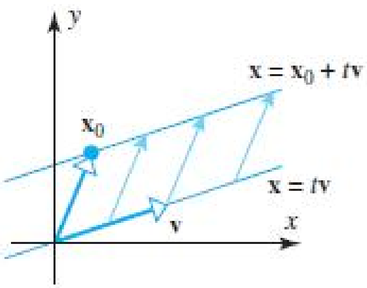


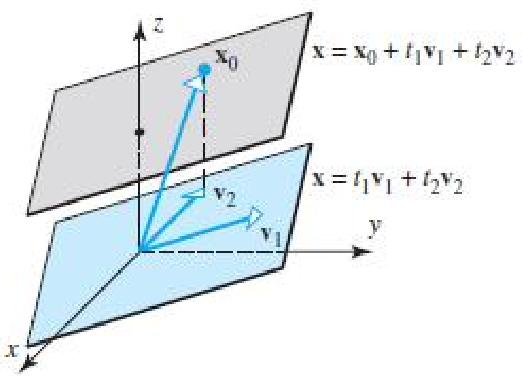
THEOREM 3.4.2 *Let W be the plane in R^3 that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by*

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{3}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{4}$$





DEFINITION 1 If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{5}$$

defines the ***line through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

DEFINITION 2 If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{6}$$

defines the ***plane through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}_1$ ***and*** $\mathbf{v}_2$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

5 Cross Product

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{1}$$

Cross Products and Dot Products

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (*Lagrange's identity*)

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (*relationship between cross and dot products*)

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ (*relationship between cross and dot products*)

Properties of Cross Product

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

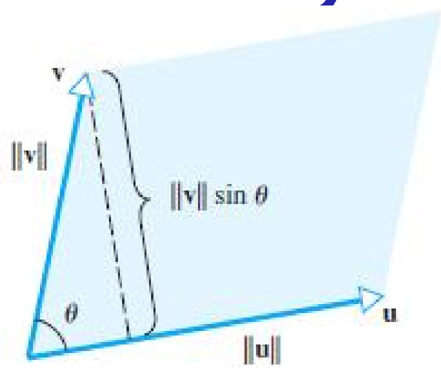
(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Geometry of the Cross Product



$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$$

THEOREM 3.5.3 Area of a Parallelogram

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in 3-space, then* $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.

Geometry of Determinants

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*

- Some of these slides have been adapted/modified in part/whole from the following textbook:

  Howard Anton & Chris Rorres. (2000). Elementary Linear Algebra. New York: John Wiley & Sons, Inc