Systems of Linear Equations

R.Rosnawati

Linear Equations

- Any straight line in xy-plane can be represented algebraically by an equation of the form: $ax + by = c$

- General form: define a linear equation in the *n* variables $x_1, x_2, \ldots, x_n$:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  - Where $a_1, a_2, \ldots, a_n,$ and b are real constants.
  - The variables in a linear equation are sometimes called unknowns.

Example 1
Linear Equations

- The equations $x + 3y = 7,\ y = \frac{1}{2}x + 3z + 1,$ and $x_1 - 2x_2 - 3x_3 + x_4 = 7$ are linear.
- Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions. $\frac{1}{x} + y = 5$
- The equations sin x +cos x = 1, log x = 2

  are *not* linear.

Example 2
Finding a Solution Set

- Find the solution of $\text{(a) } 4x - 2y = 1$

- Solution(a)

  we can assign an arbitrary value to x and solve for y , or choose an arbitrary value for y and solve for x .If we follow the first approach and assign x an arbitrary value ,we obtain

  $$x = t_1,\ y = 2t_1 - \frac{1}{2} \quad \text{or} \quad x = \frac{1}{2}t_2 + \frac{1}{4},\ y = t_2$$

  - arbitrary numbers $t_{1,}\ t_2$ are called parameter.
  - for example

  $$t_1 = 3 \text{ yields the solution } x = 3, y = \frac{11}{2} \quad \text{as} \quad t_2 = \frac{11}{2}$$

Solution

- A solution of a linear equation is a sequence of n numbers $s_1, s_2, \ldots, s_n$ such that the equation is satisfied. The set of all solutions of the equation is called its solution set or general solution of the equation

Example
Finding a Solution Set

- Find the solution of (b) $x_1 - 4x_2 + 7x_3 = 5.$

- Solution(b)

  we can assign arbitrary values to any two variables and solve for the third variable.

  - for example

    $$x_1 = 5 + 4s - 7t, \qquad x_2 = s, \qquad x_3 = t$$

  - where s, t are arbitrary values

Linear Systems

- A finite **set** of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations or a linear system .

- A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a solution of the system.

- A system has **no** solution is said to be inconsistent ; if there is **at least** one solution of the system, it is called consistent.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

↑ An arbitrary system of m linear equations in n unknowns

Linear Systems

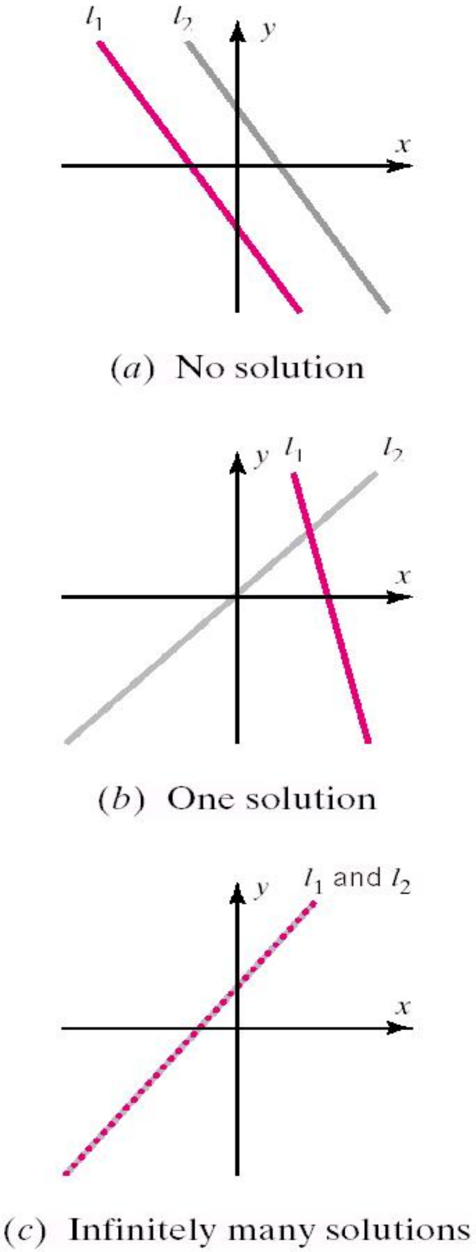

Figure 1.1.1

- *Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.*
- A general system of two linear equations: (Figure1.1.1)

$$a_1 x + b_1 y = c_1 \; (a_1, b_1 \text{ not both zero})$$

$$a_2 x + b_2 y = c_2 \; (a_2, b_2 \text{ not both zero})$$

  - Two lines may be parallel -> <u>no solution</u>
  - Two lines may intersect at only one point -> <u>one solution</u>
  - Two lines may coincide -> <u>infinitely many solution</u>

Augmented Matrices

- The location of the +'s, the x's, and the ='s can be abbreviated by writing only the rectangular array of numbers.
- This is called the augmented matrix for the system.
- Note: must be written in the same order in each equation as the unknowns and the constants must be on the right.

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{array}
$$

1th column

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & b_m
\end{array}\right]
$$

1th row

Elementary Row Operations

- The basic method for solving a system of linear equations is to replace the given system by **a new system that has the same solution set** but which is **easier** to solve.

- Since the **rows** of an augmented matrix correspond to the **equations** in the associated system. new systems is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically. These are called elementary row operations.

  1. Multiply an equation through by an nonzero constant.
  2. Interchange two equation.
  3. Add a multiple of one equation to another.

Example 3
Using Elementary row Operations

$$\begin{aligned} x + \quad y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$\xrightarrow{\text{add -2 times the first equation to the second}}$

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$\xrightarrow{\text{add -3 times the first equation to the third}}$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{\text{add -2 times the first row to the second}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{\text{add -3 times the first row to the third}}$$

Example 3
Using Elementary row Operations

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

multiply the second equation by $\frac{1}{2}$ $\longrightarrow$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= 0 \end{aligned}$$

add -3 times the second equation to the third $\longrightarrow$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

multily the second row by $\frac{1}{2}$ $\longrightarrow$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right]$$

add -3 times the second row to the third $\longrightarrow$

Example 3
Using Elementary row Operations

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Multiply the third equation by -2 $\longrightarrow$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second equation to the first $\longrightarrow$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multily the third row by -2 $\longrightarrow$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second row to the first $\longrightarrow$

Example 3
Using Elementary row Operations

$$
\begin{aligned}
x \qquad + \tfrac{11}{2} z &= \tfrac{35}{2} \\
y - \tfrac{7}{2} z &= -\tfrac{17}{2} \\
z &= 3
\end{aligned}
\quad \xrightarrow{\substack{\text{Add } -\frac{11}{2} \text{ times} \\ \text{the third equation} \\ \text{to the first and } \frac{7}{2} \text{ times} \\ \text{the third equation} \\ \text{to the second}}} \quad
\begin{aligned}
x &= 1 \\
y &= 2 \\
z &= 3
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}
\xrightarrow{\substack{\text{Add } -\frac{11}{2} \text{ times} \\ \text{the third row} \\ \text{to the first and } \frac{7}{2} \\ \text{times the third row} \\ \text{to the second}}}
\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}
$$

- The solution x=1,y=2,z=3 is now evident.